Constructing Hamiltonian Circuits

When all nodes have degree of at least n/2

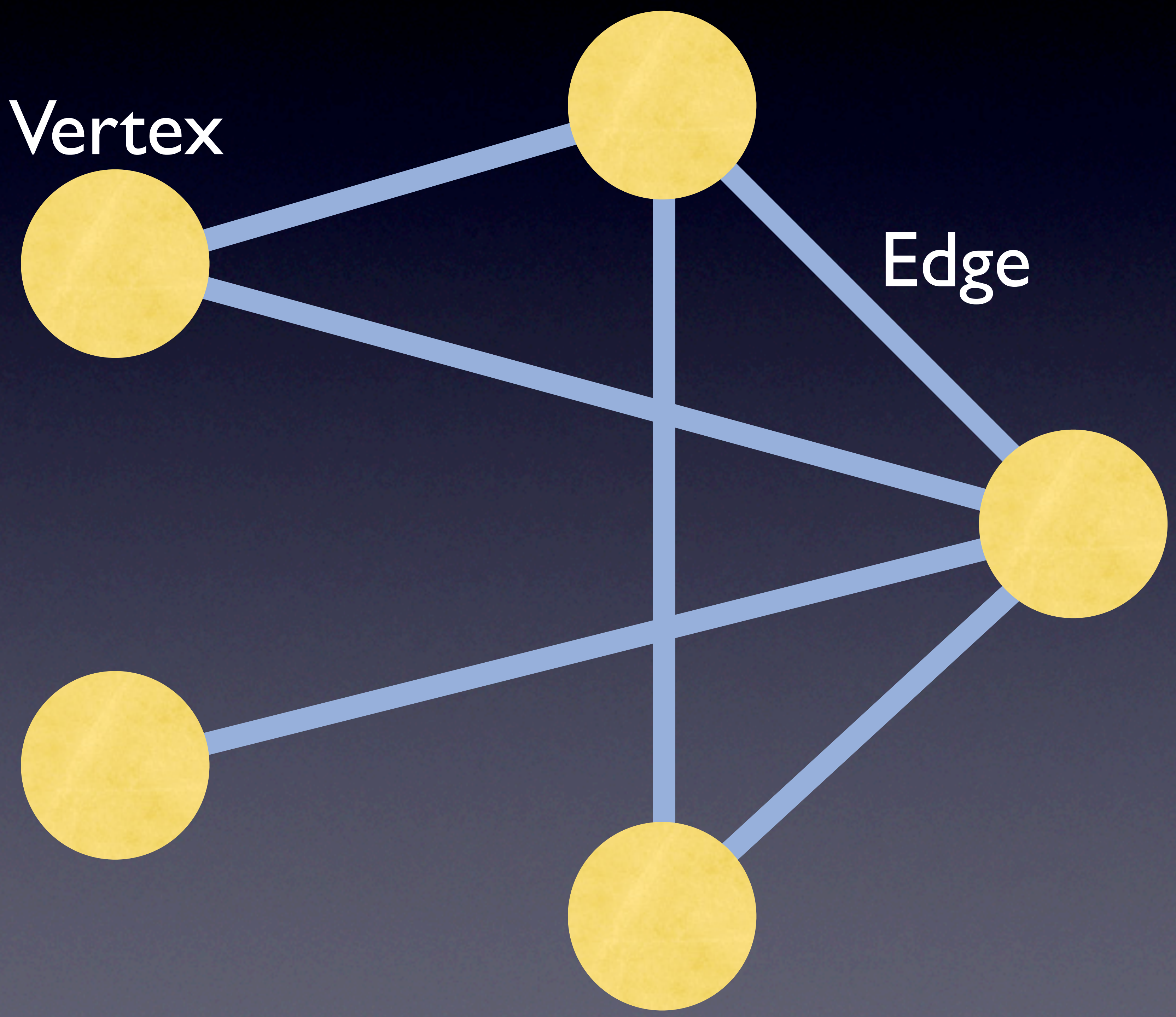
(Also: an implementation in C++ using Boost)

Presented by Alan Hogan @ SUnMaRC, February 2008

Graphs

- A graph is a collection of vertices (or points) and edges (which connect the vertices).

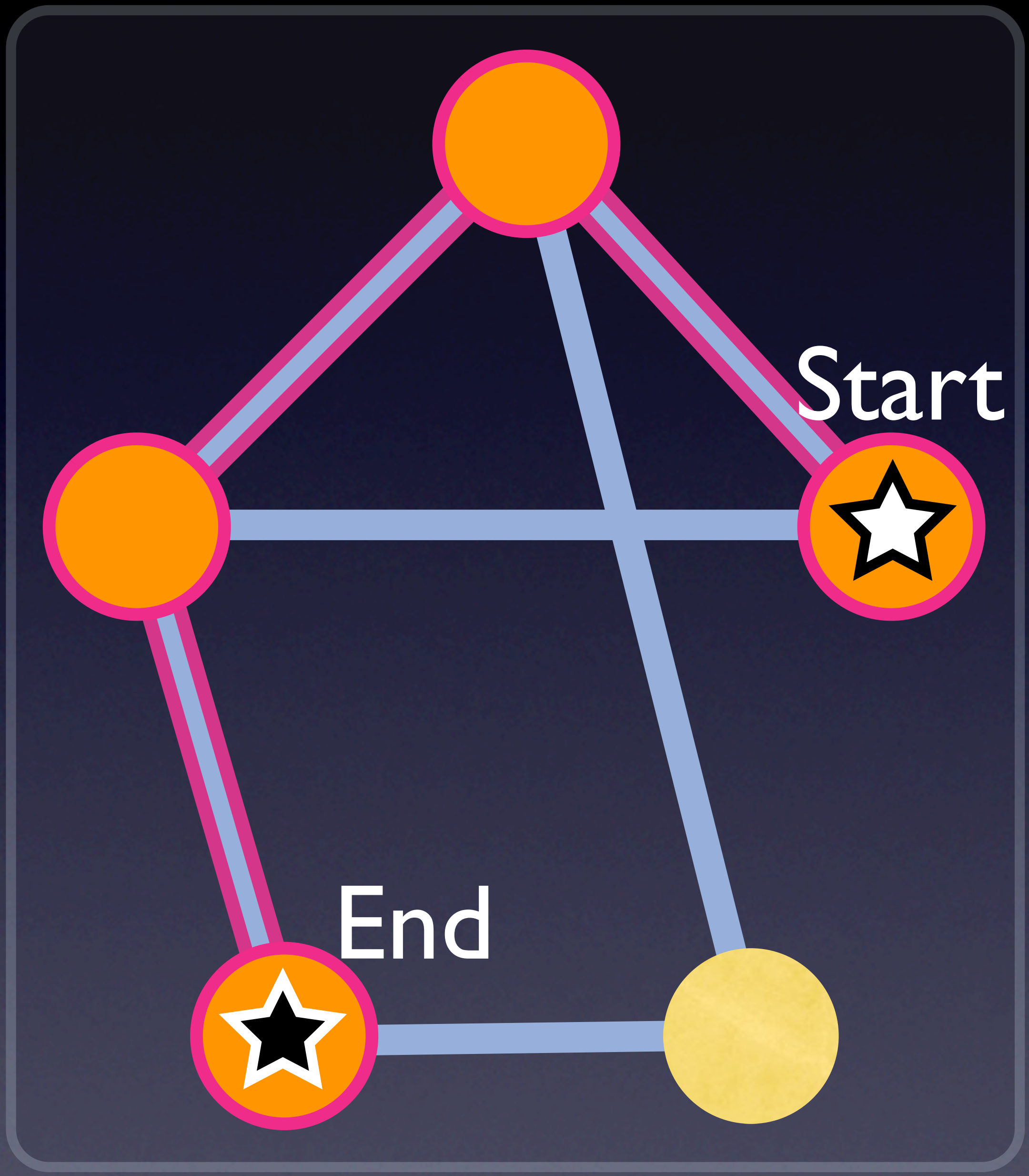
Example Graph



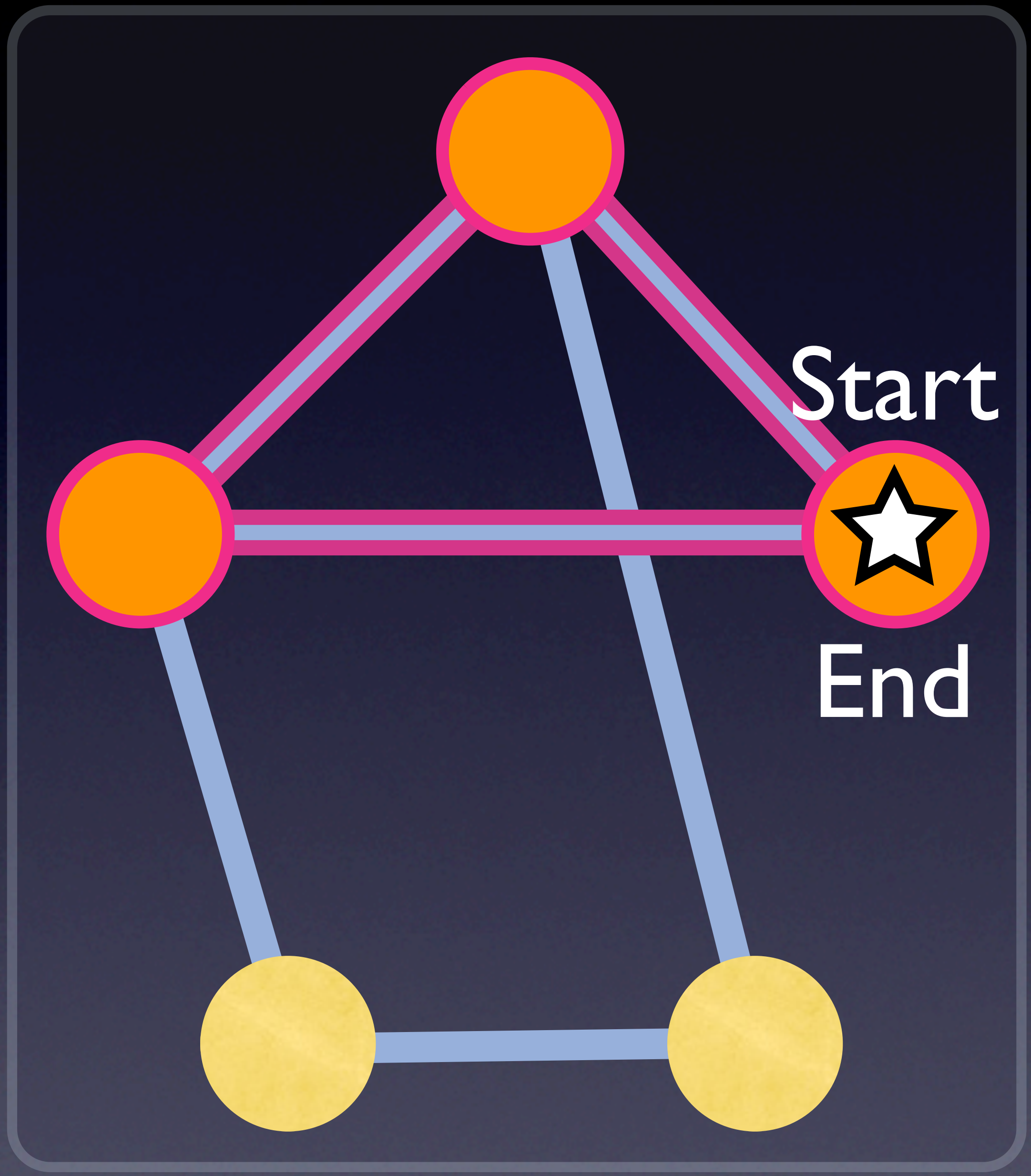
Paths and Circuits

- Paths are series of vertices connected by edges
- A circuit is a closed path (starts & ends at the same vertex)

Path



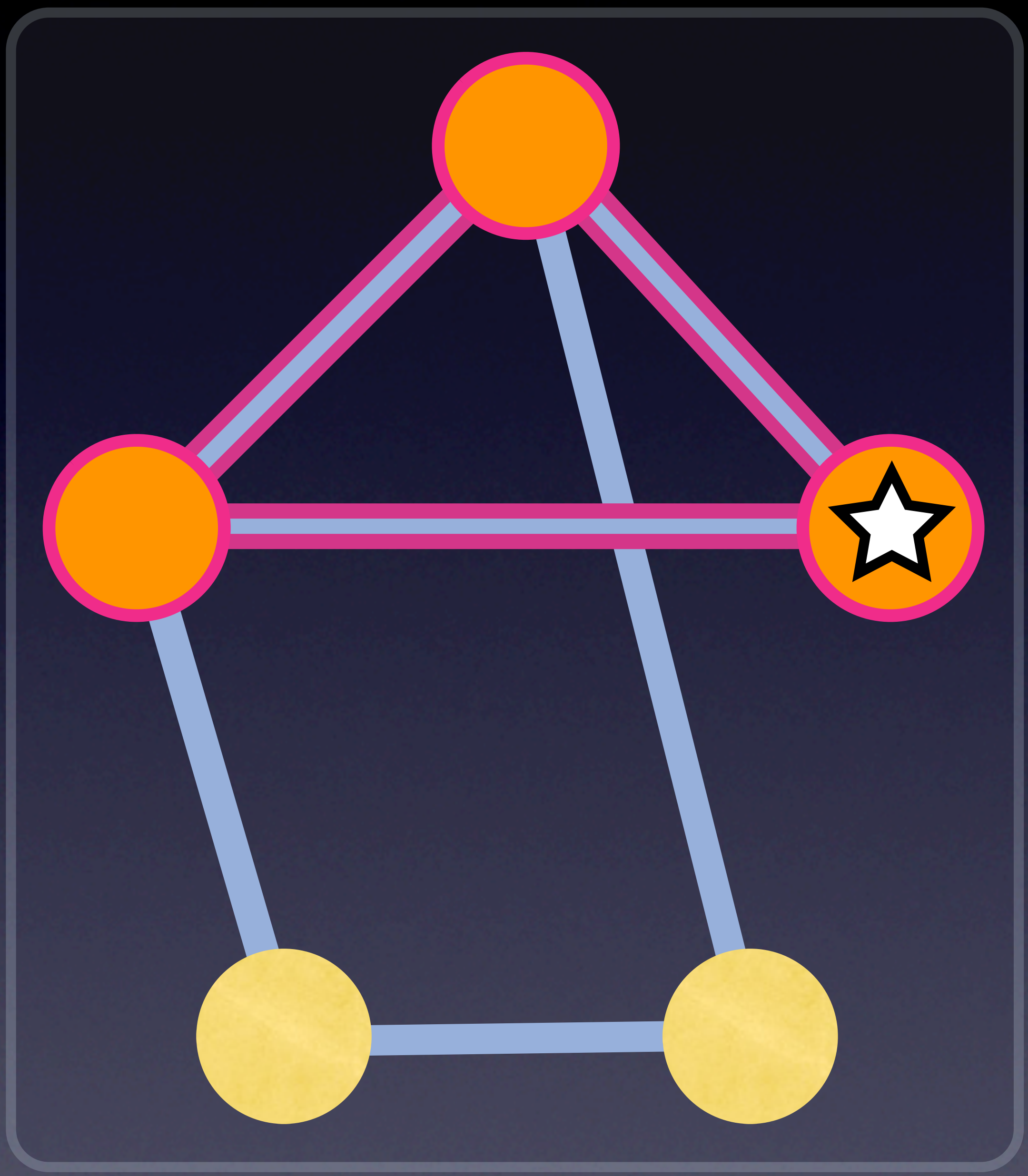
Circuit



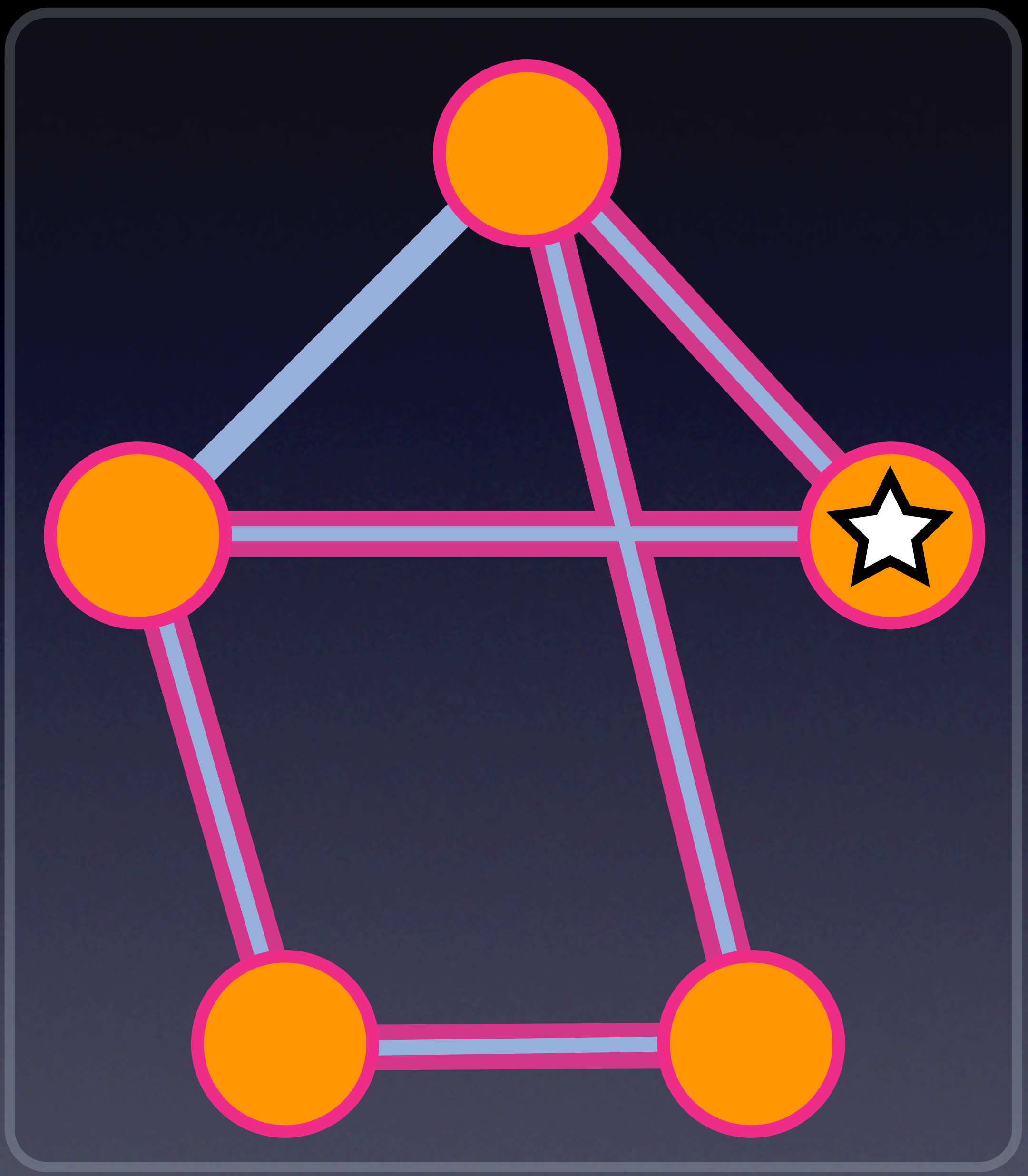
Hamiltonian Circuits

- A Hamiltonian circuit is a closed path which visits every vertex in the graph exactly one time
- Also called "Hamiltonian Cycles"

Plain Circuit



Hamiltonian



Problem

- General algorithms to find Hamiltonian circuits are **slow**, running in non-polynomial time — it is an NP-complete problem
- We can use an efficient algorithm, however, in some cases, thanks to Dirac and Ore...

Dirac's Theorem (1952)

- A simple graph with n vertices ($n > 2$) is Hamiltonian if each vertex has degree $n/2$ or greater.
- (sufficient but not necessary)

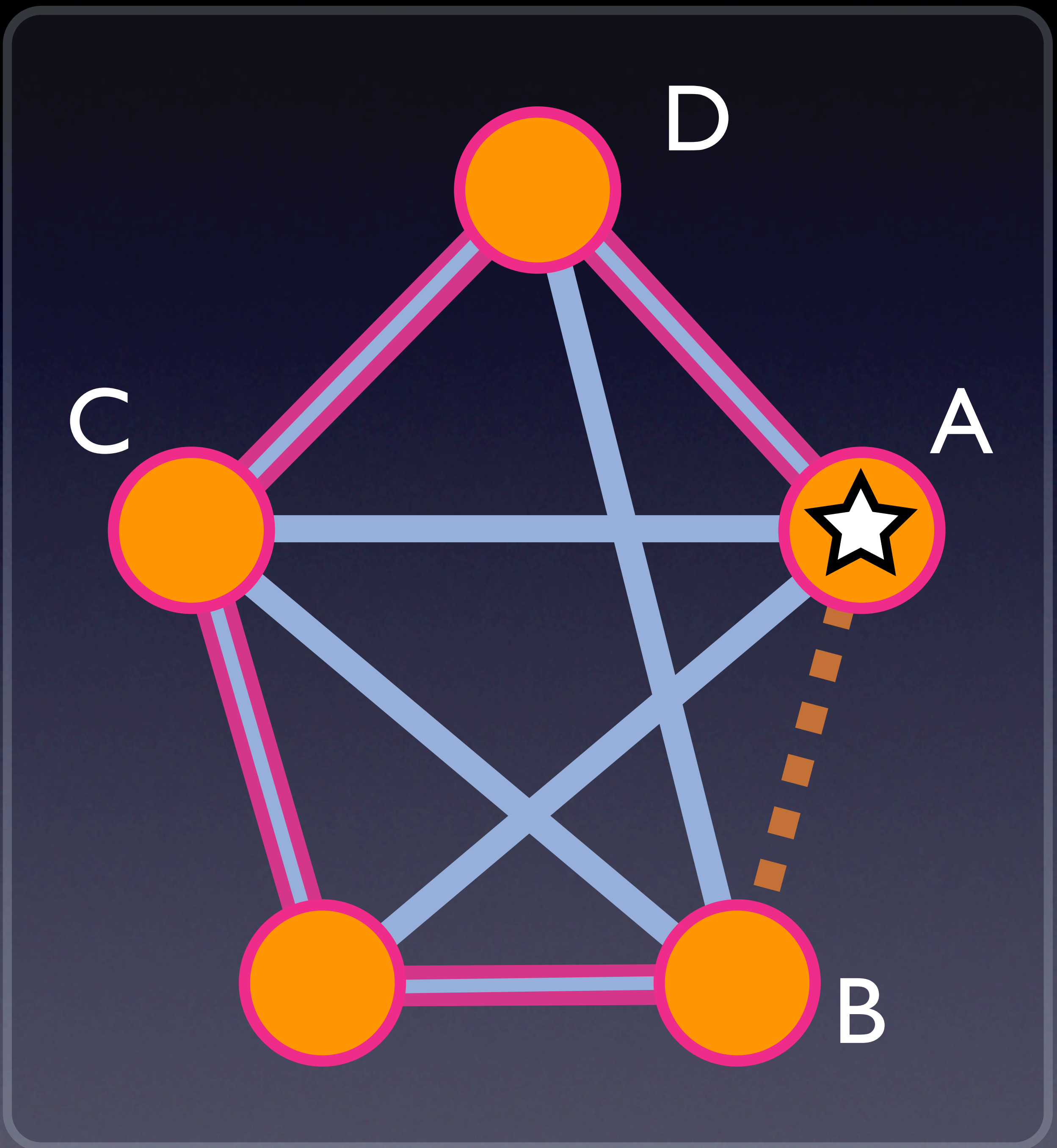
Ore's Theorem (1960)

- Generalization of Dirac's Theorem
- If G is a simple graph with n vertices, where $n \geq 3$, and if for each pair of non-adjacent vertices v and w, $\deg(v) + \deg(w) \geq n$, then G is Hamiltonian

Also in Ore's paper...

- Ore's restatement of Dirac's principle lends itself to an interesting & useful principle
- For graphs satisfying the pre-requisite condition, an existing almost-complete Hamiltonian circuit with a gap between vertices *A* and *B* where there "should" be the final edge can be "repaired."

Using Ore’s Algorinthm



Missing edge

1. Find a two vertices *C* & *D* s.t. edges (*A*,*C*) and (*B*,*D*) exist; (*C*,*D*) is in our almost-complete circuit; and *D* lies between *C* and *A* on the partial circuit.
2. Connect vertex *A* to vertex *C*
3. Connect vertex *B* to a vertex *D*
4. Remove the edge between those two earlier vertices.

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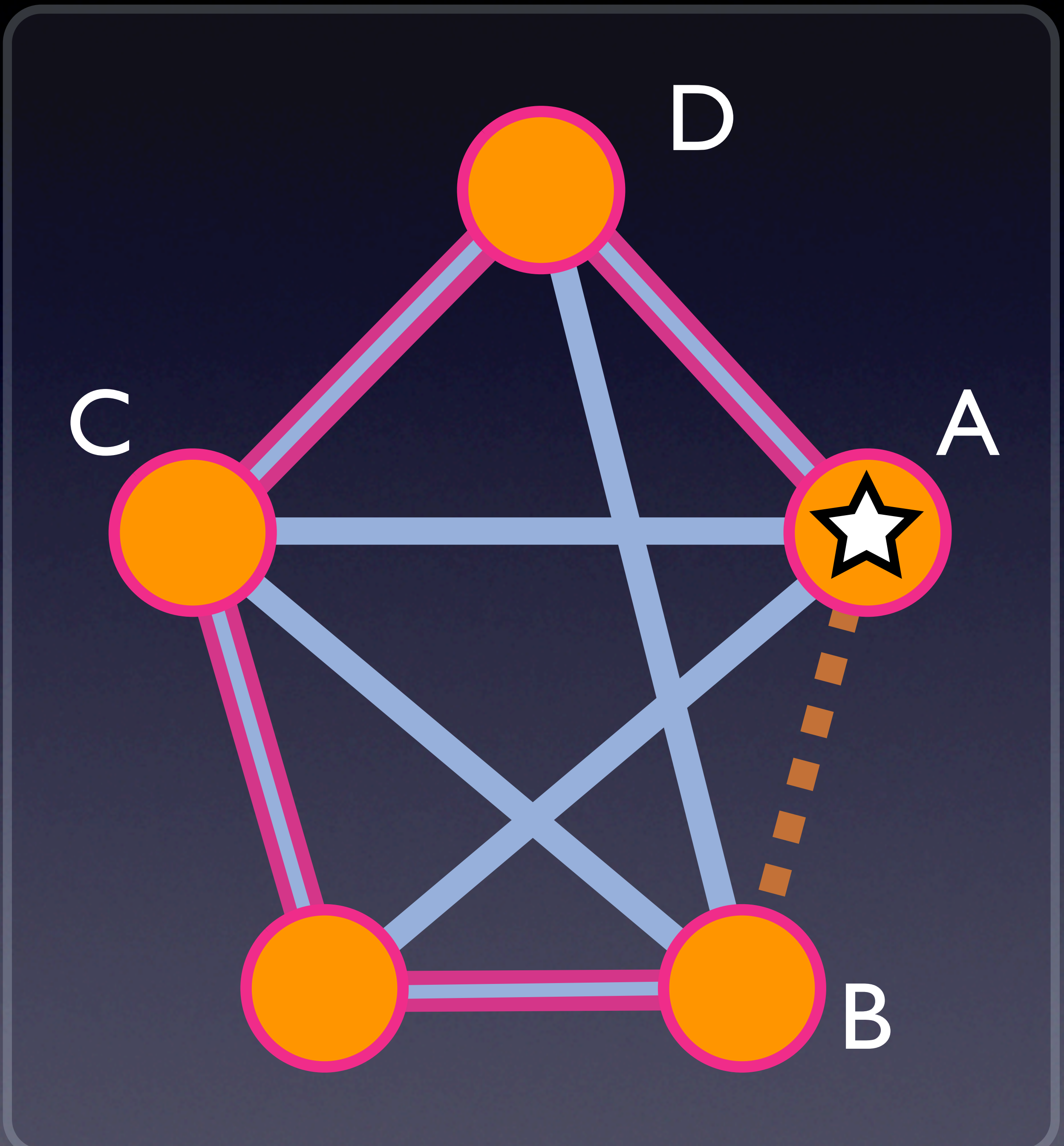

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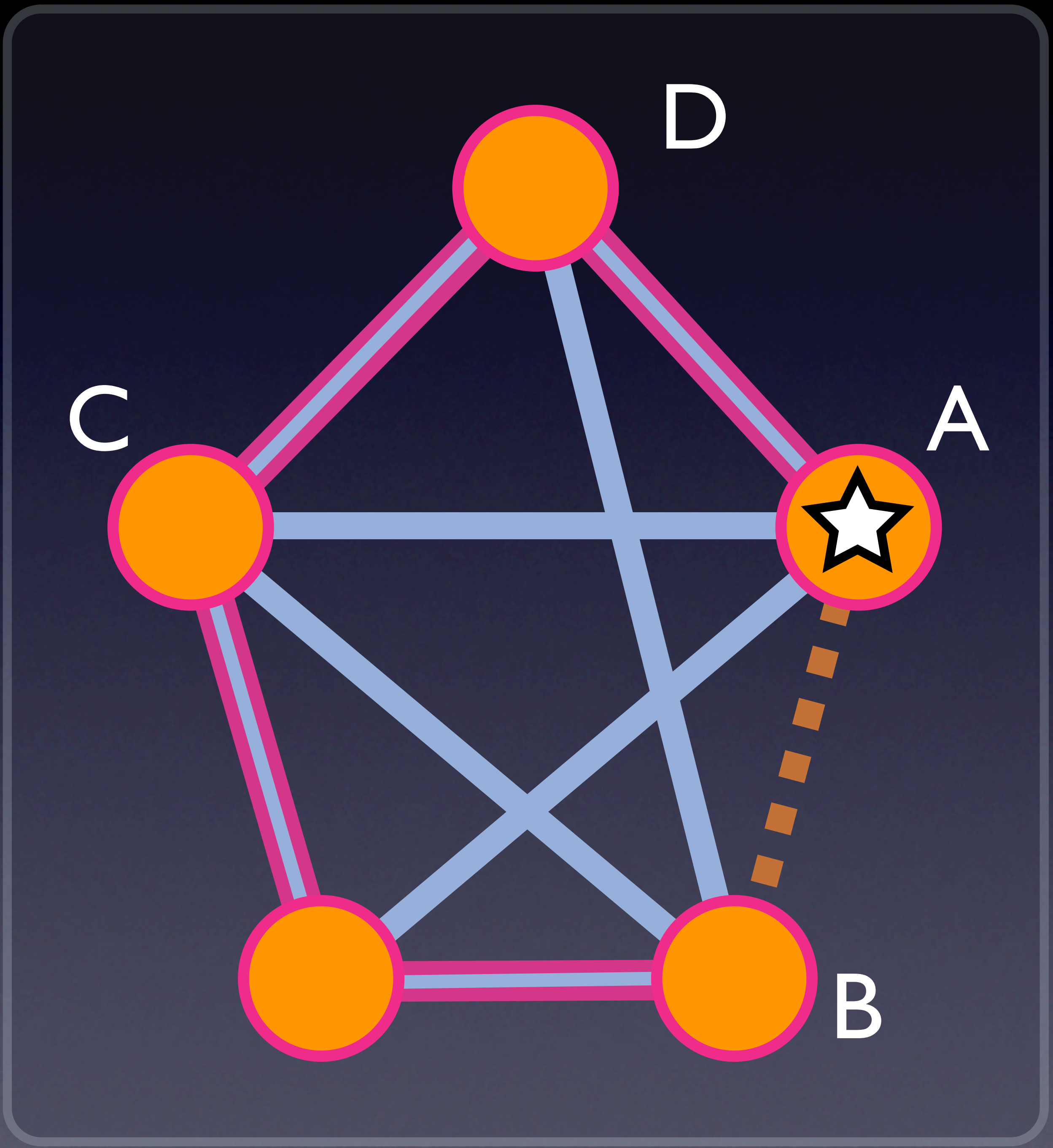

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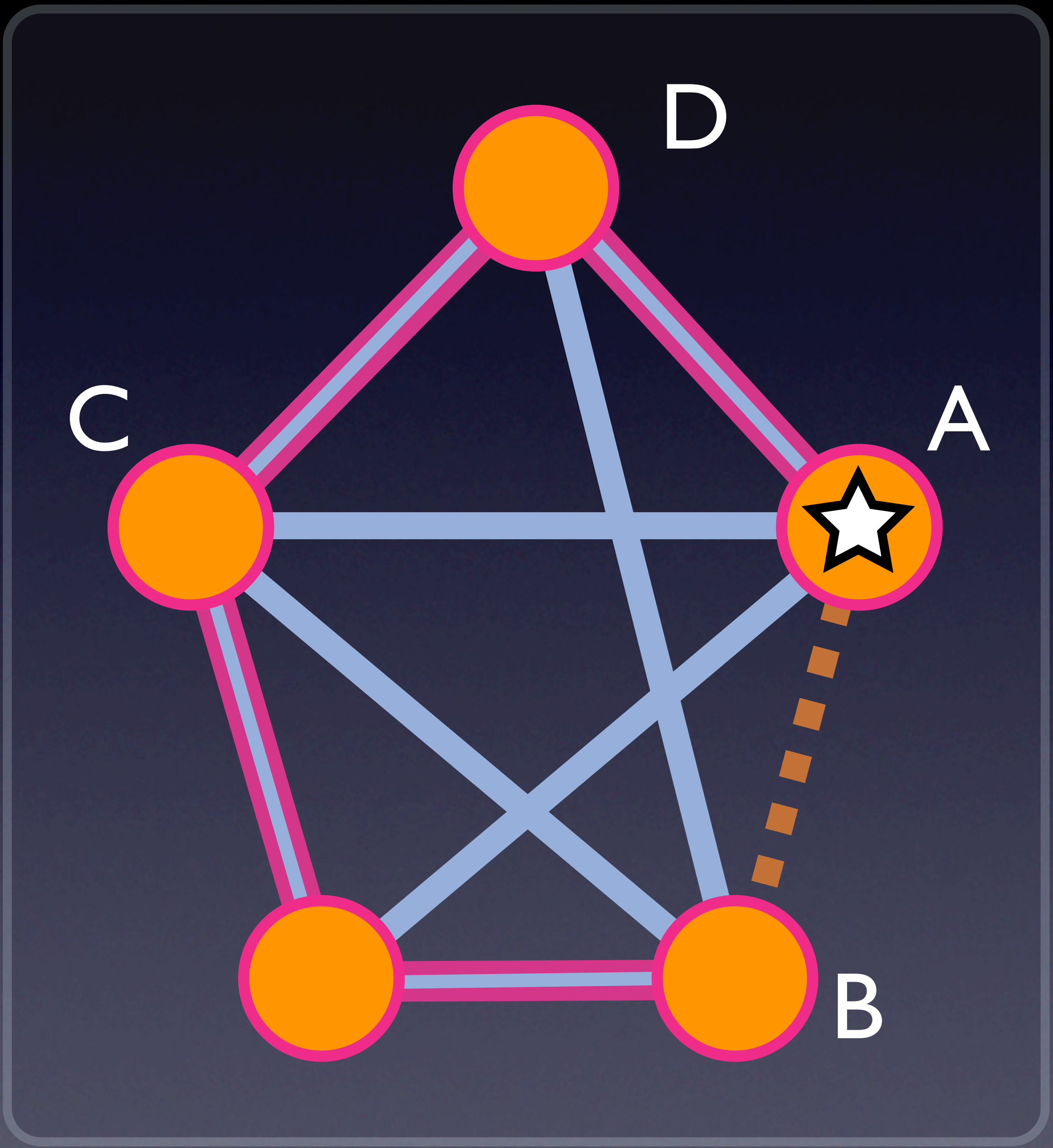

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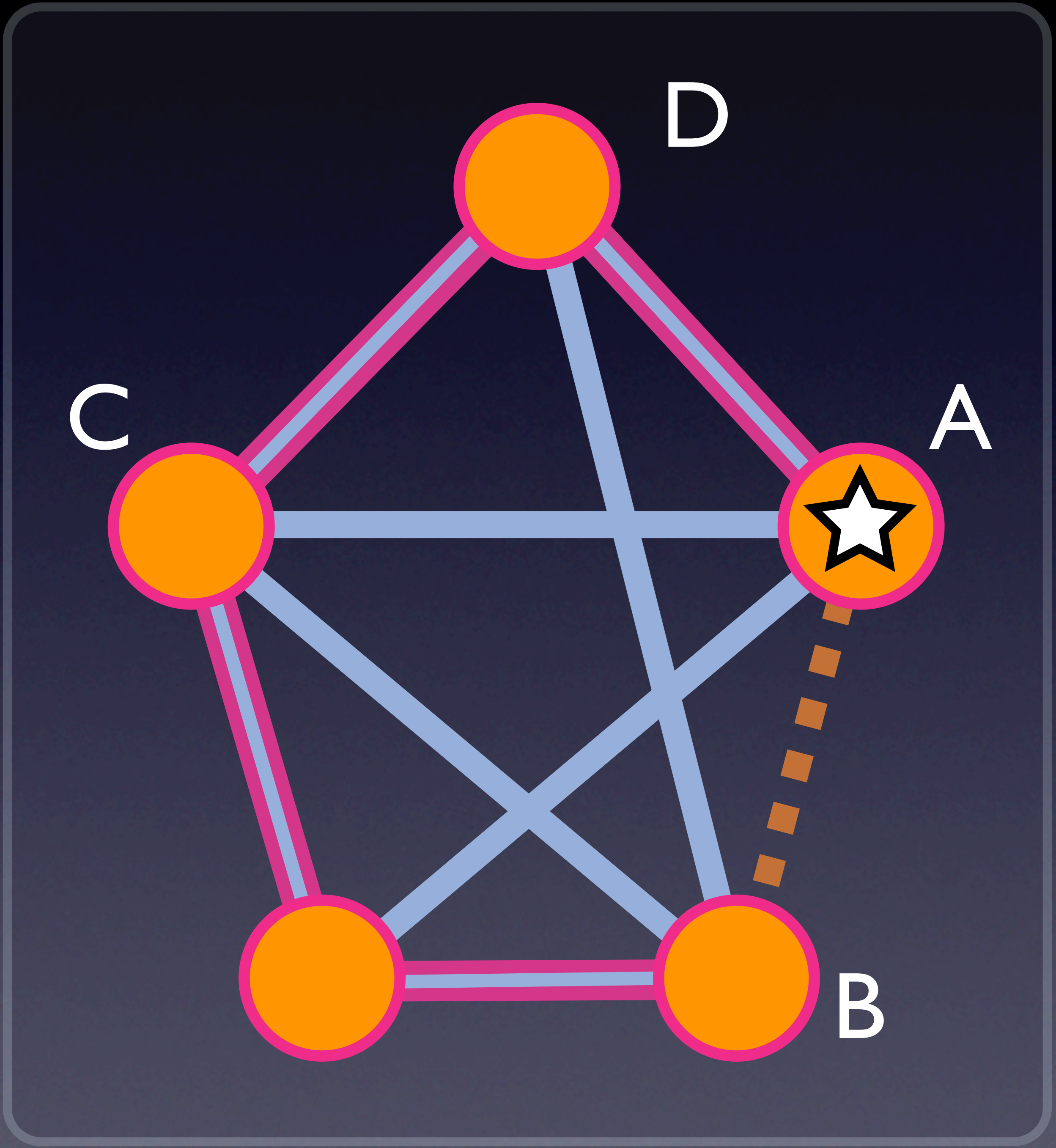

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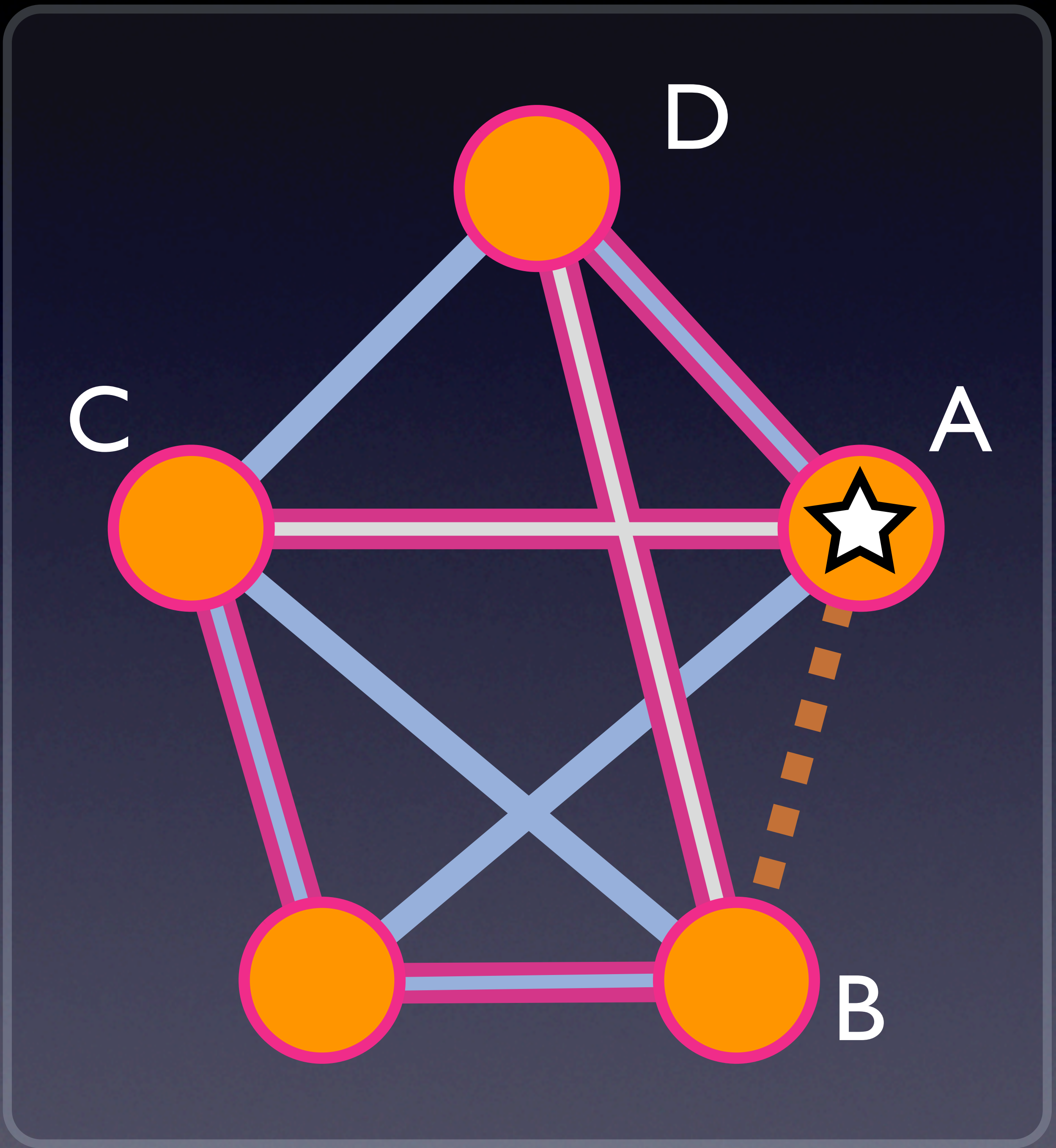


Repaired: Full circuit

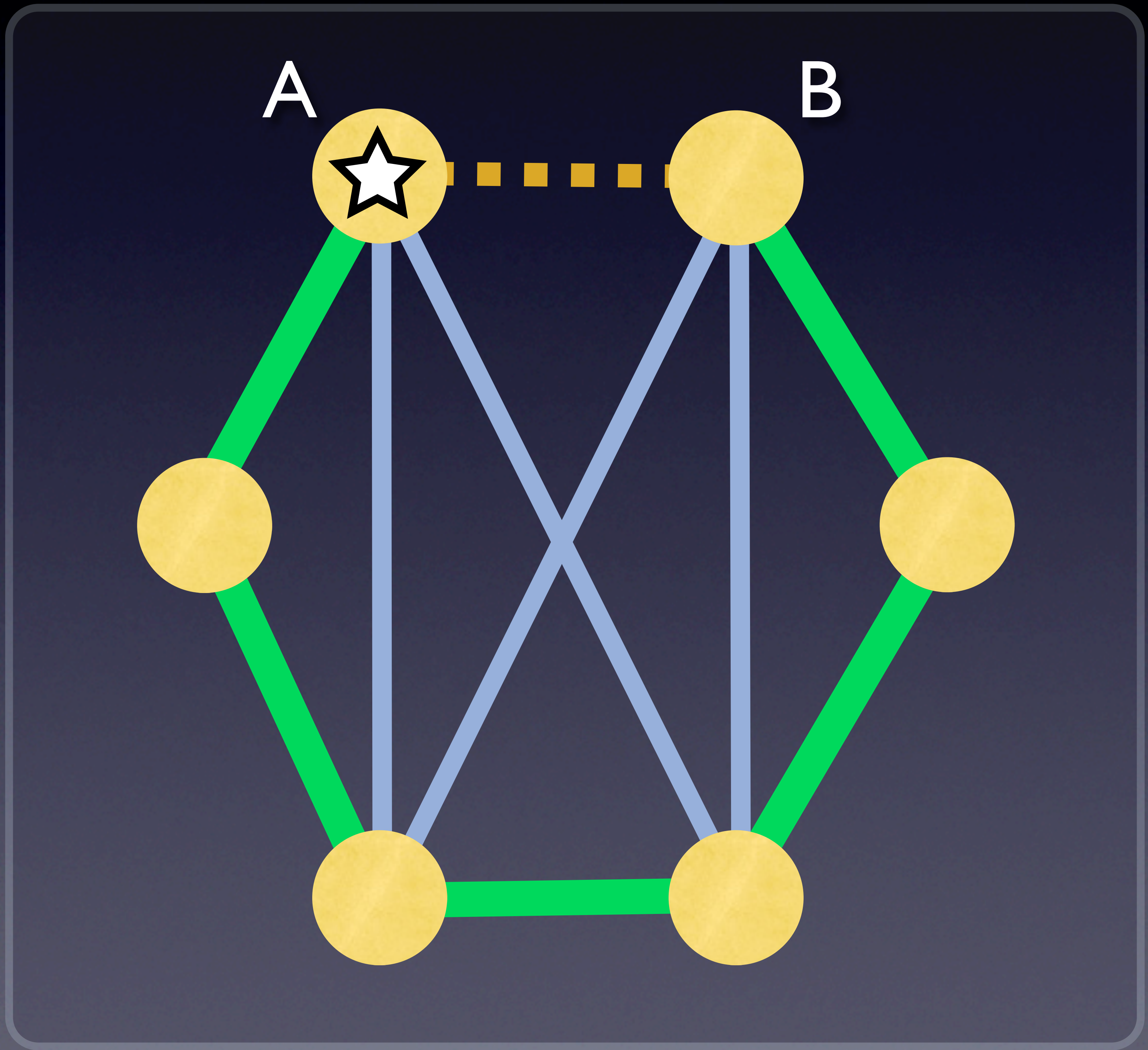
1. Find a two vertices *C* & *D* s.t. edges (*A*,*C*) and (*B*,*D*) exist; (*C*,*D*) is in our almost-complete circuit; and *D* lies between *C* and *A* on the partial circuit.
2. Connect vertex *A* to vertex *C*
3. Connect vertex *B* to a vertex *D*
4. Remove the edge between those two earlier vertices.

Why does that work?

- We know that at least one pair of such desirable contiguous earlier vertices *C* and *D* exist because each vertex has at least half as many edges as there are vertices
- Proof by the pigeonhole principle
- Boxes = potential pairs of vertices C & D = $n-3$
- Pigeons = edges from A or B = $2(n/2-1) = n-2$

Worst case

(Edges not connected to *A* or *B* and not on the circuit are not depicted)



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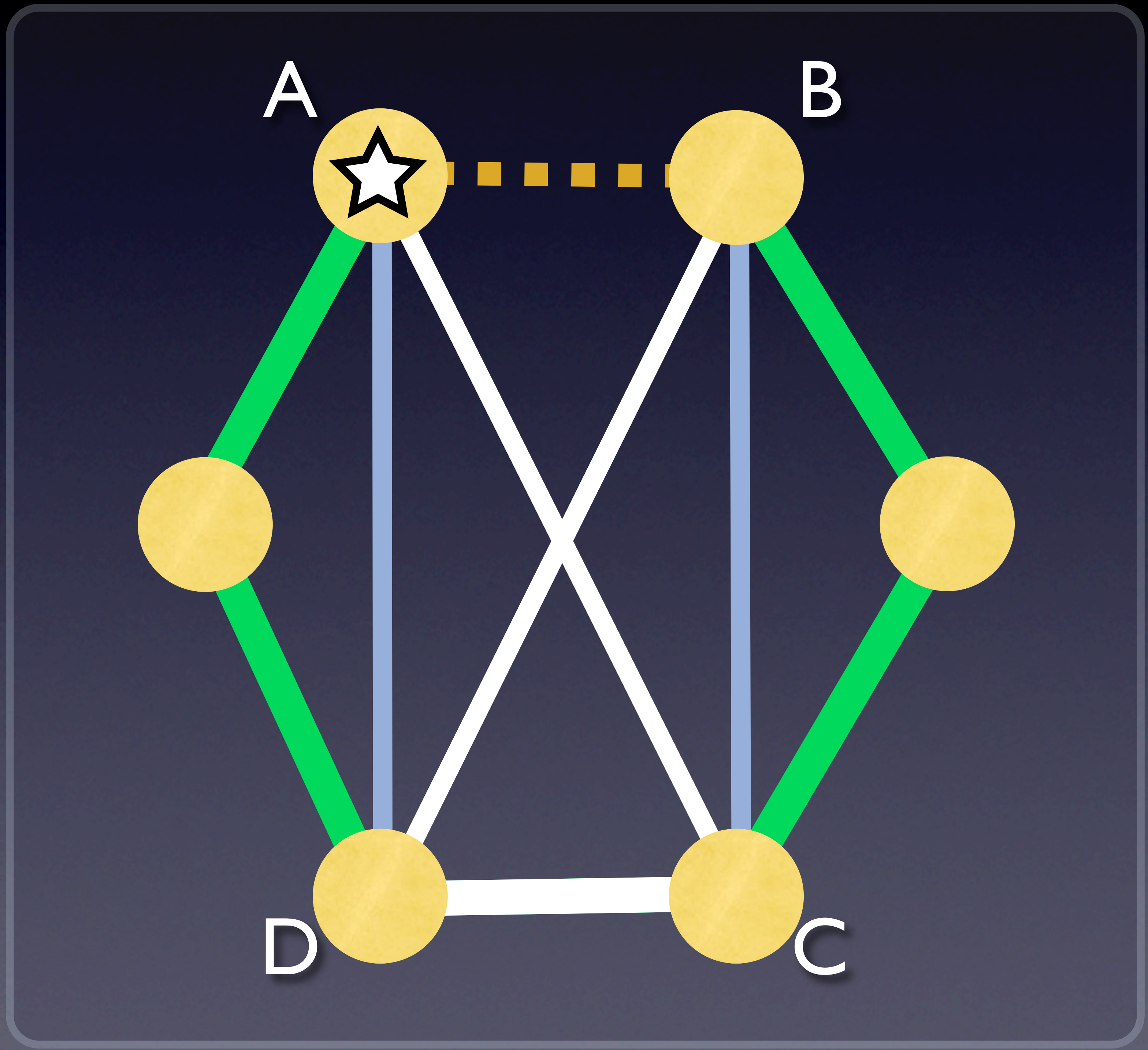

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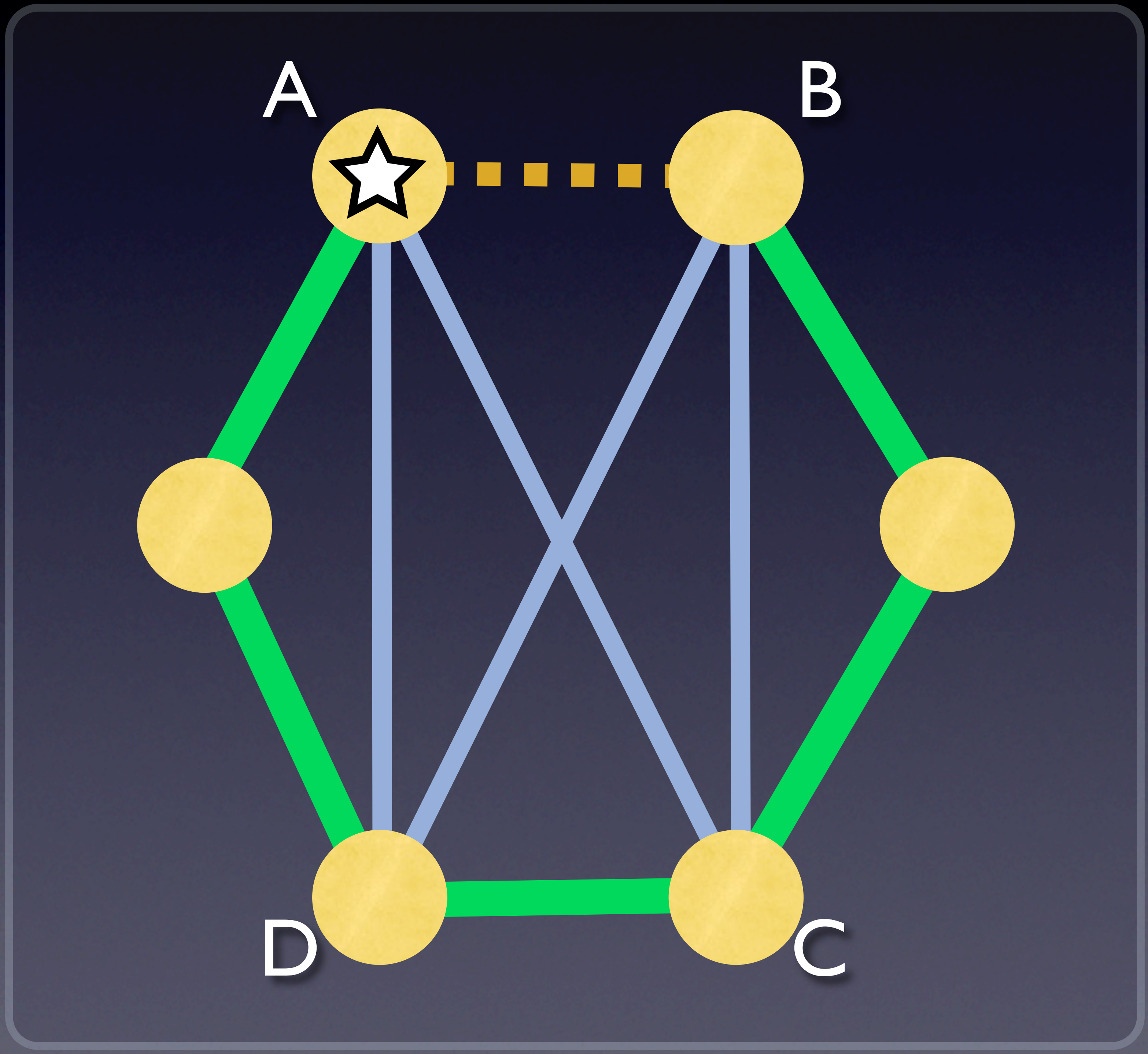
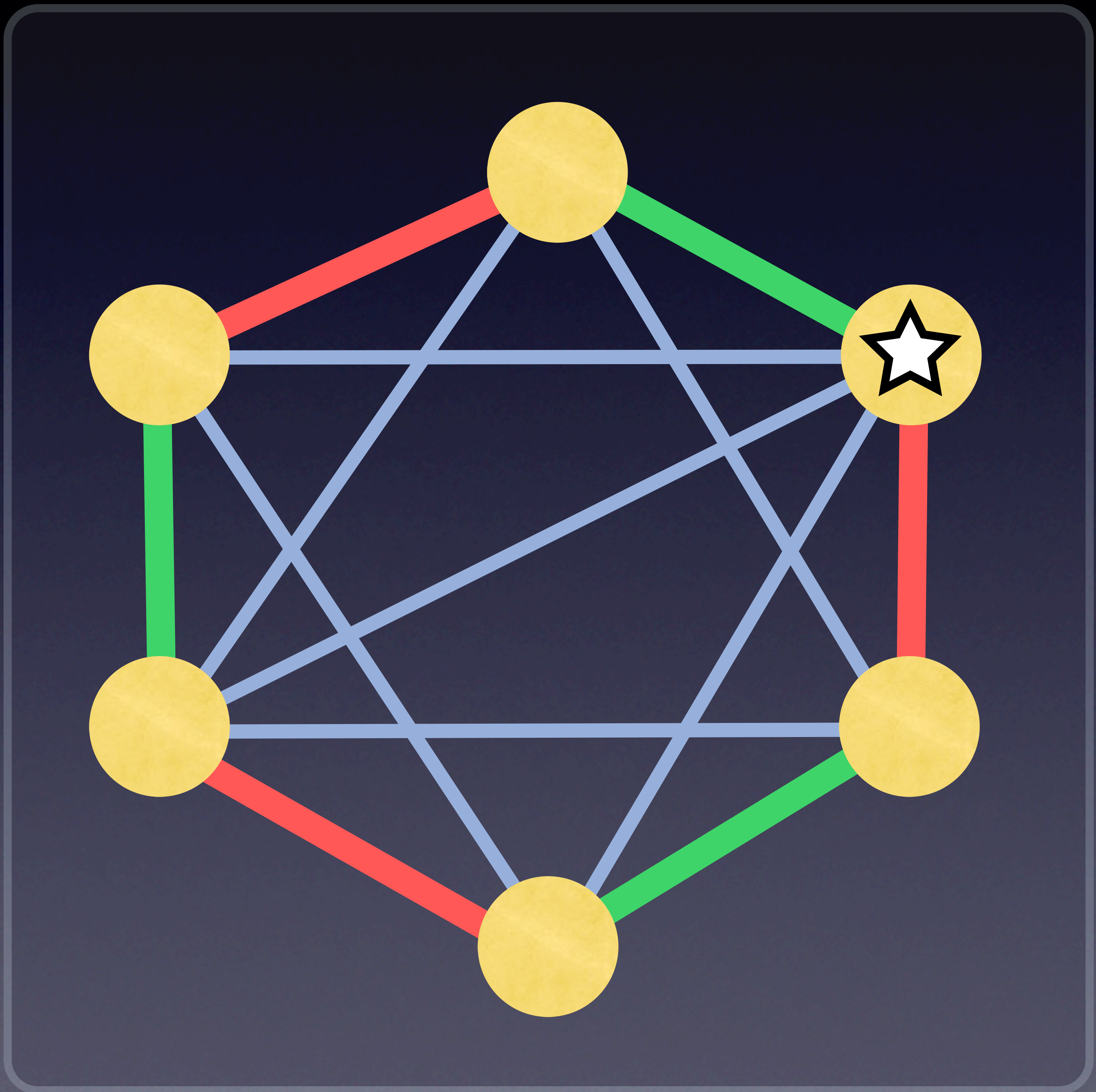


Repeated Use

- Repeated use of the algorithm suggested by Ore's paper allows us to find a Hamiltonian circuit for any graph in our scope (all vertices have at least $n/2$ edges)

Repeating the Algorithm



1. Pretend we have a circuit
2. Acknowledge one pretend edge does not really exist
3. Fix that edge. We have a pretend circuit again, but it's closer to true
4. Go back to step 2. Repeat until all edges really exist

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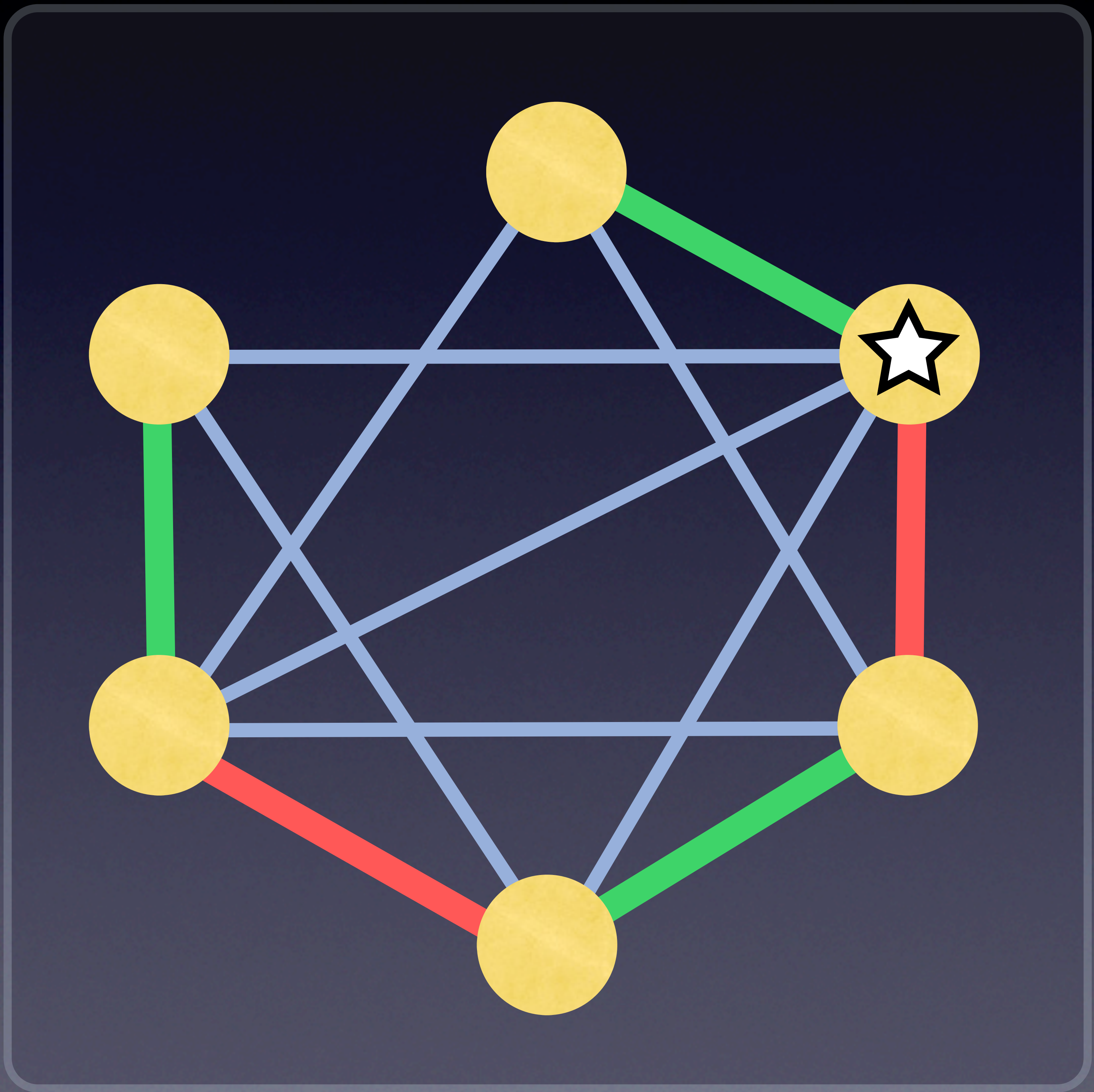

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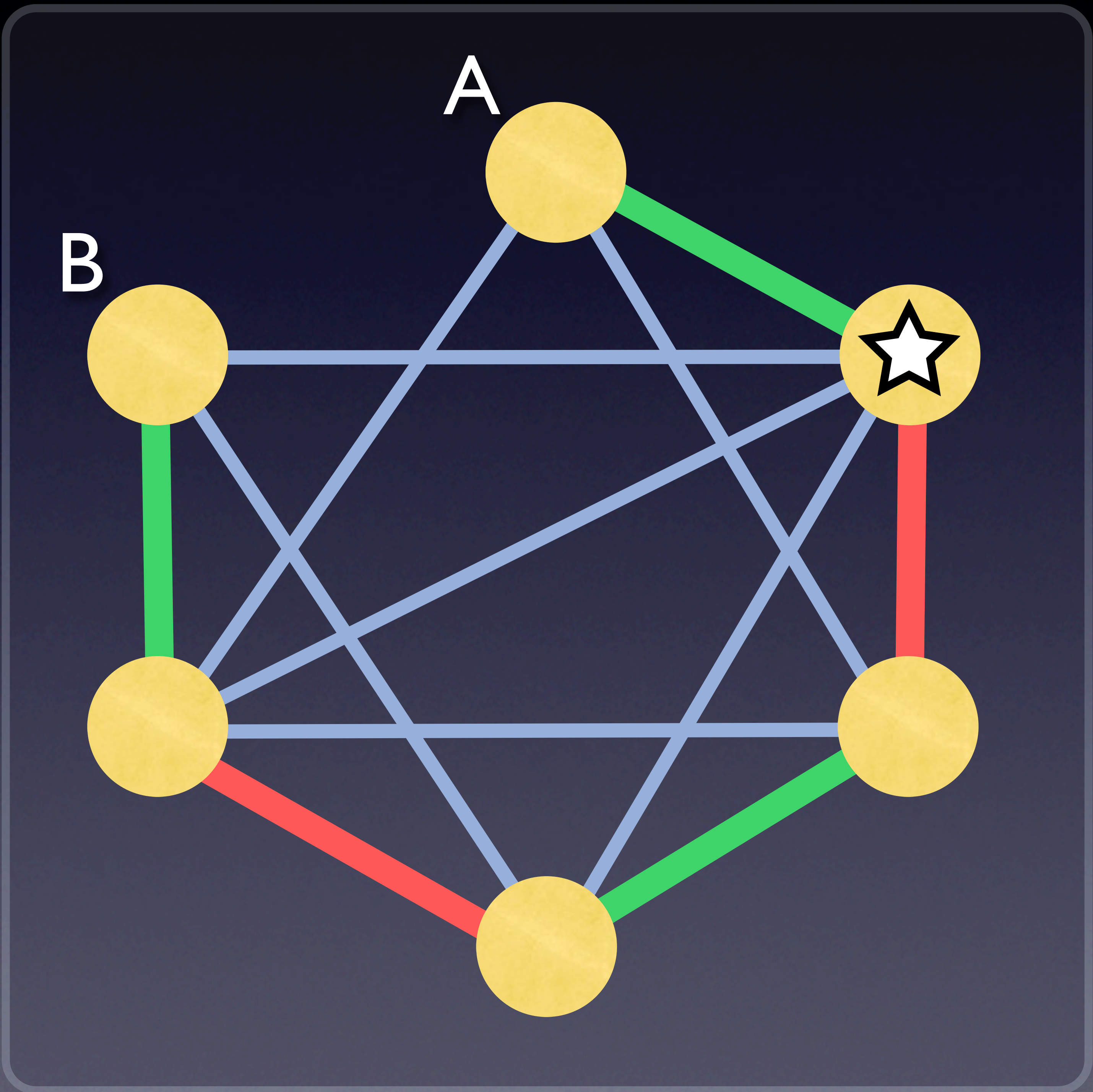

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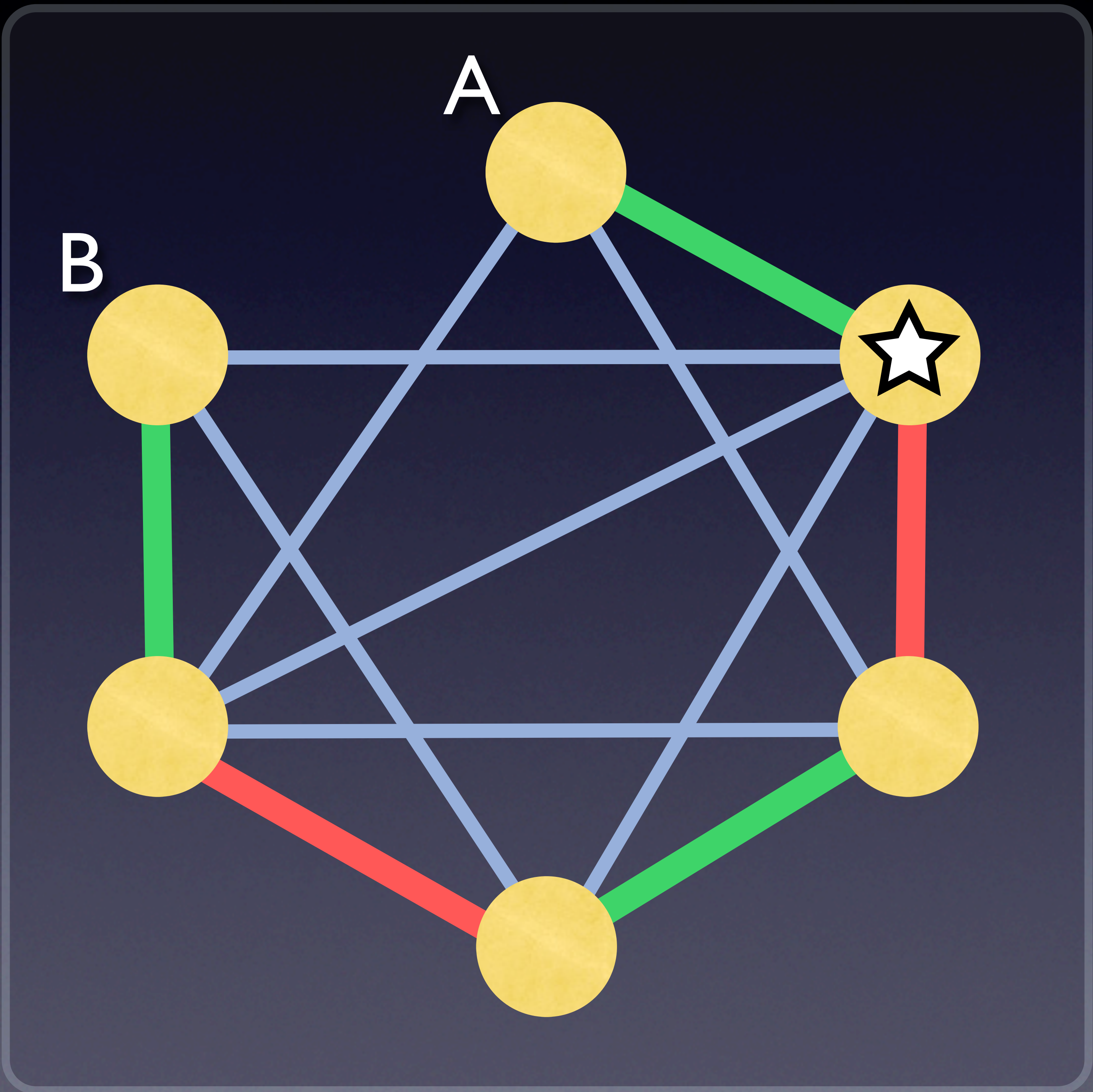

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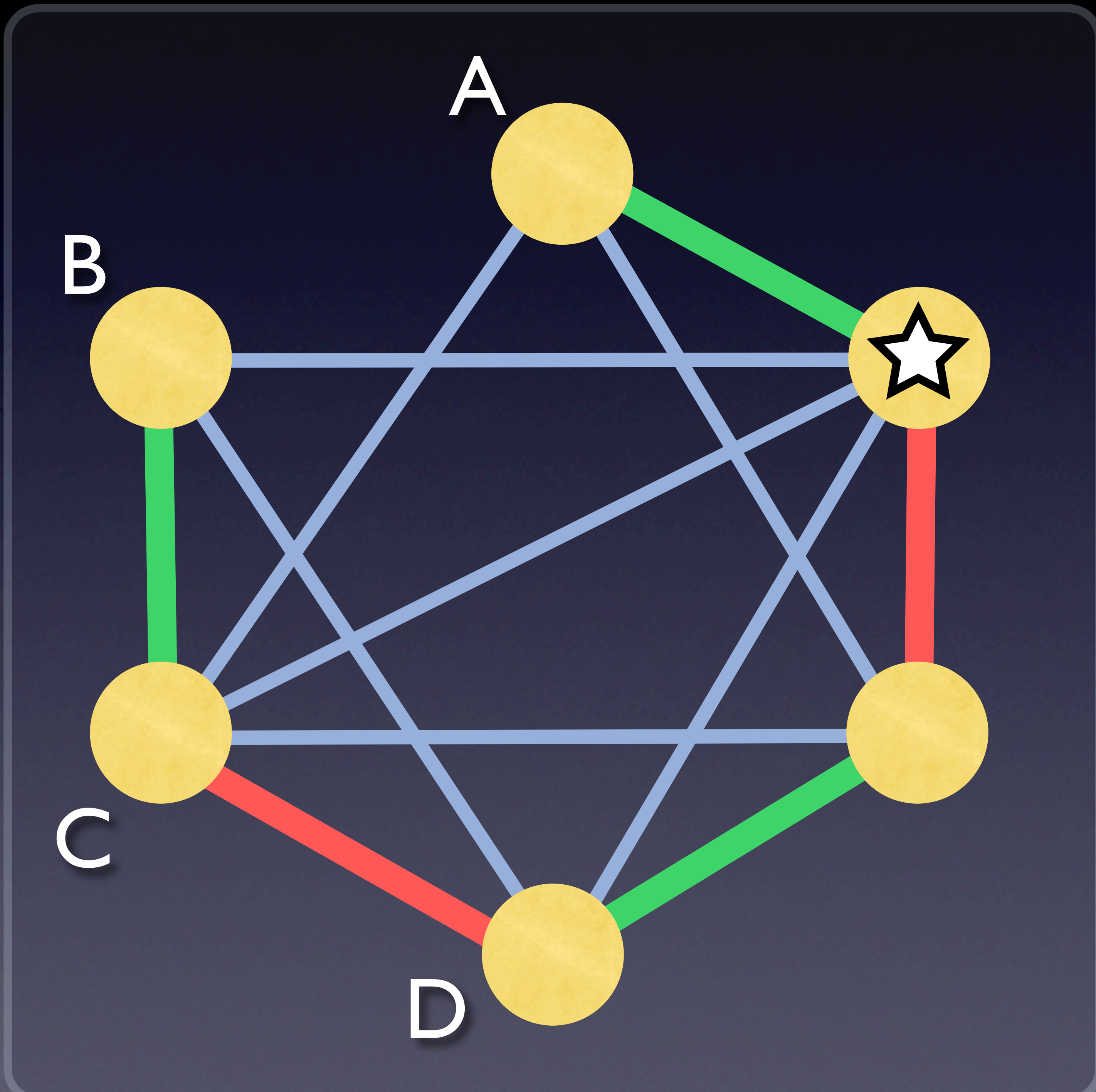

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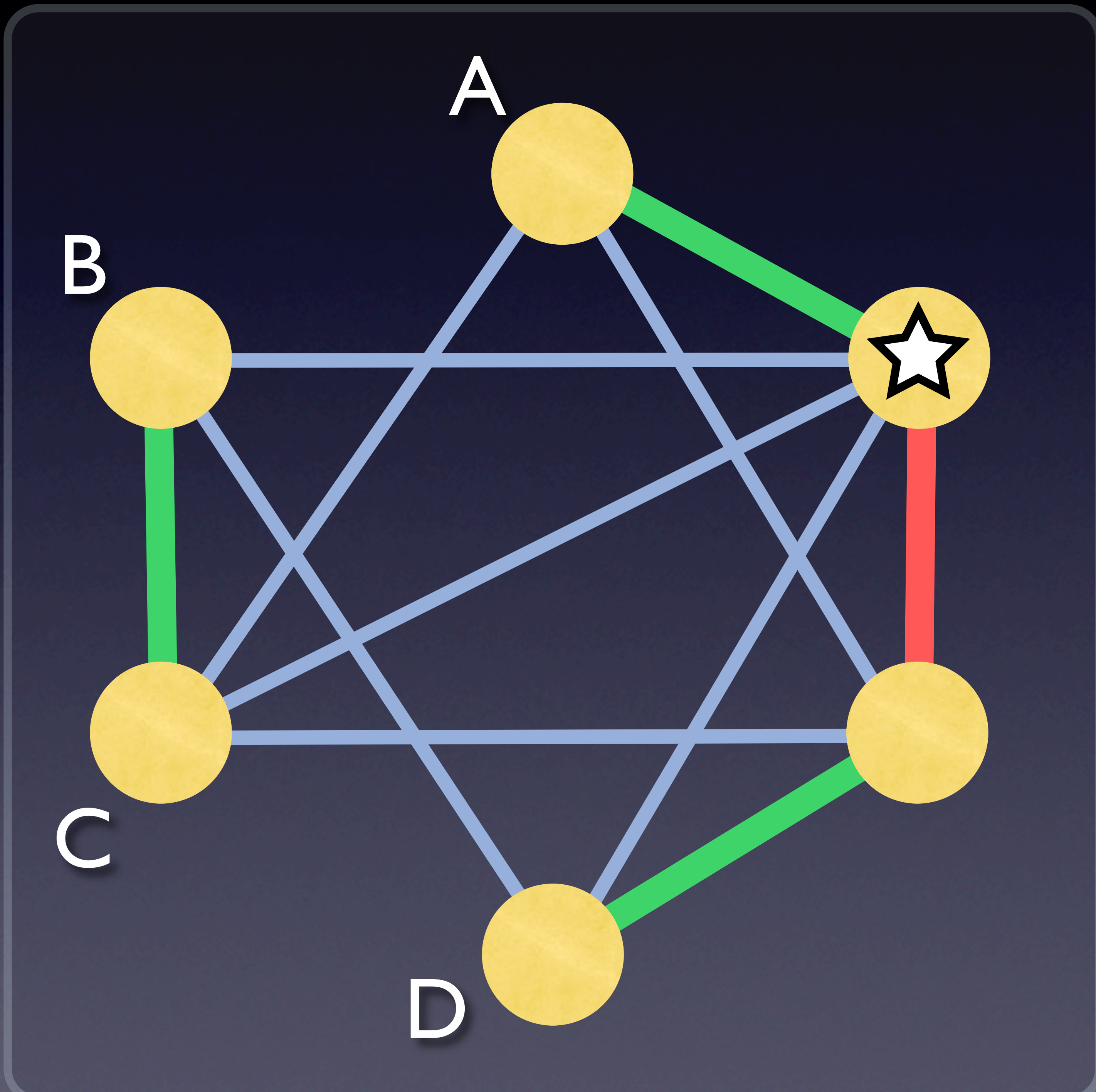

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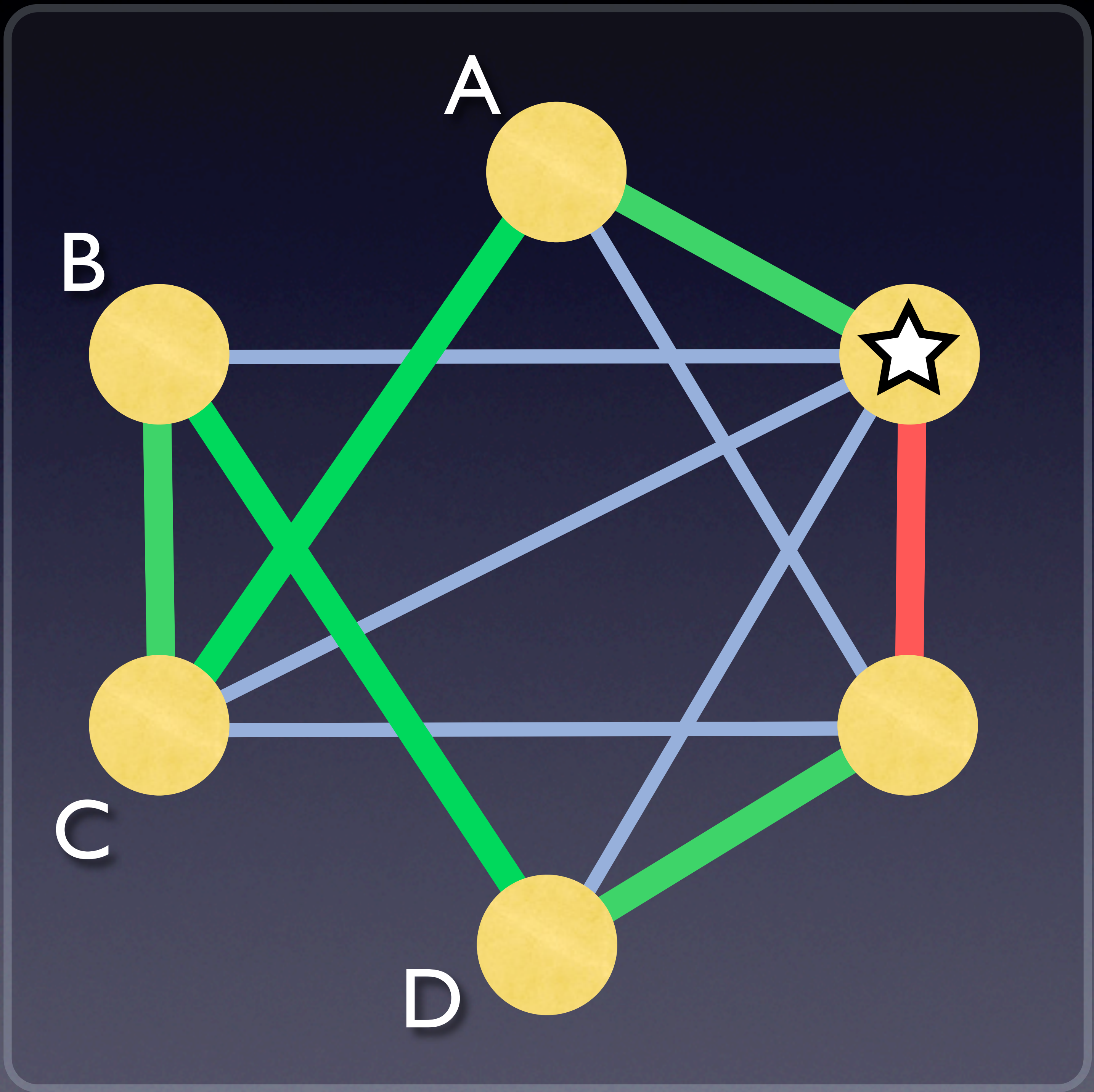

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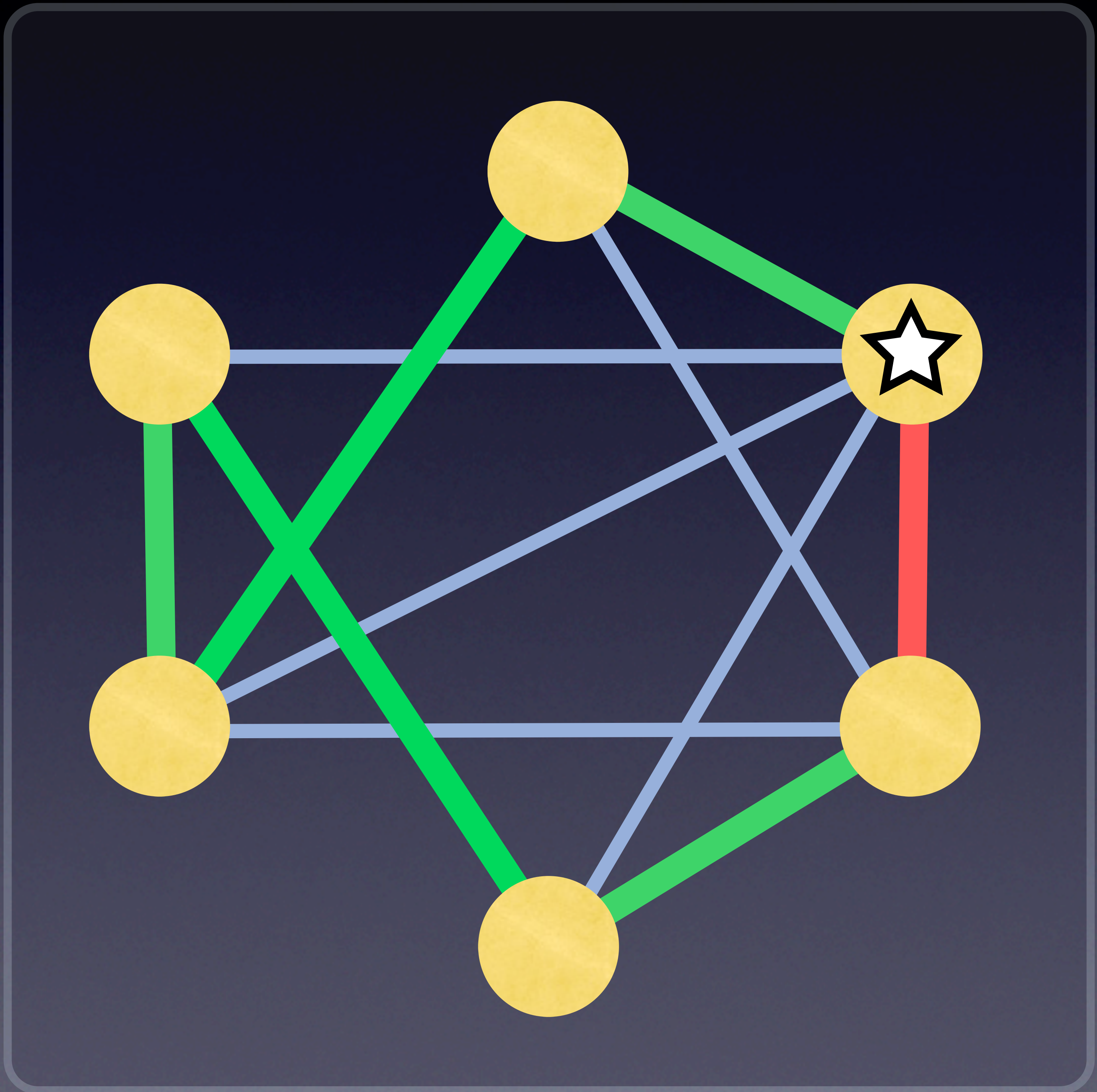

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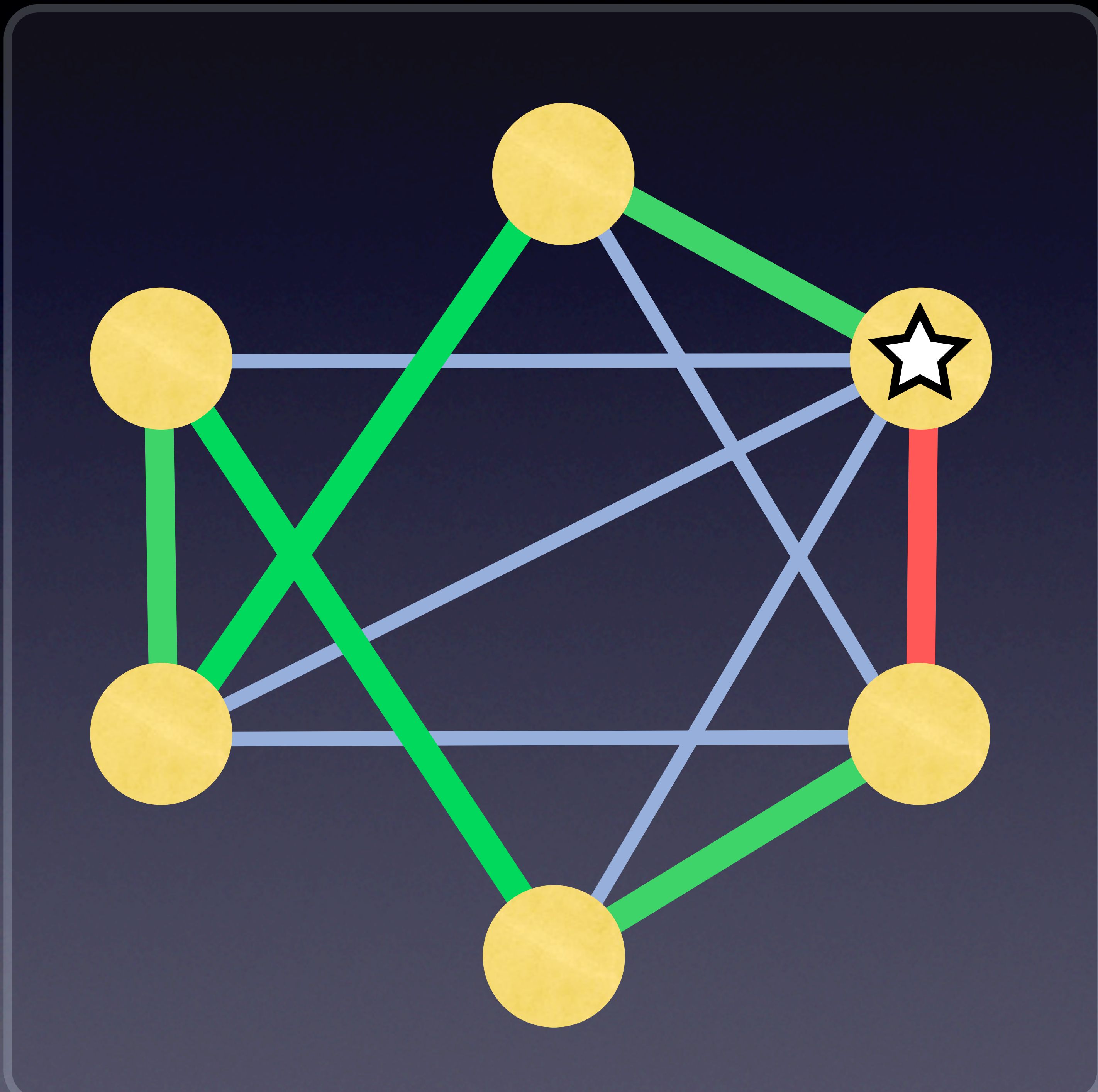

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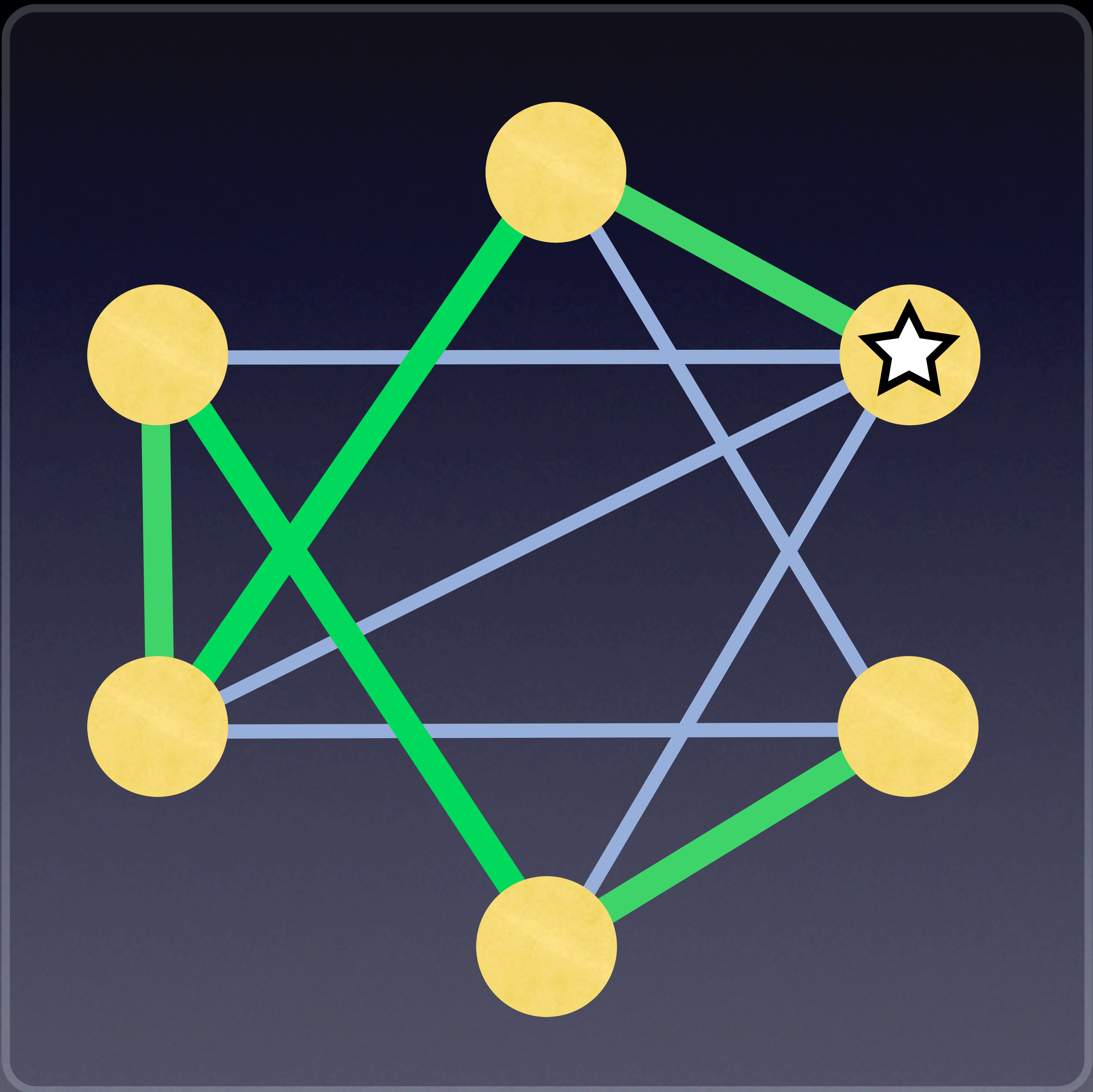

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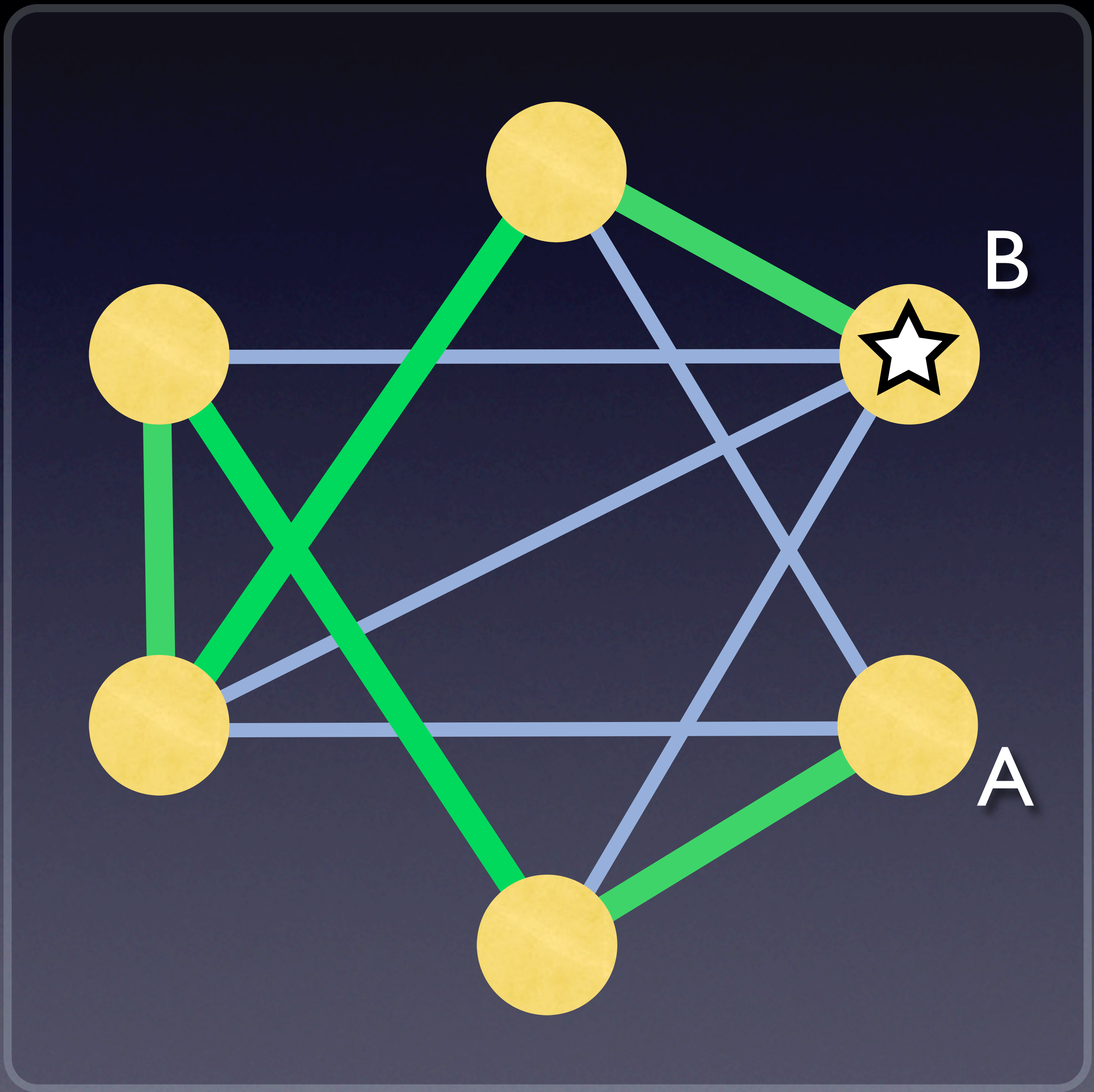

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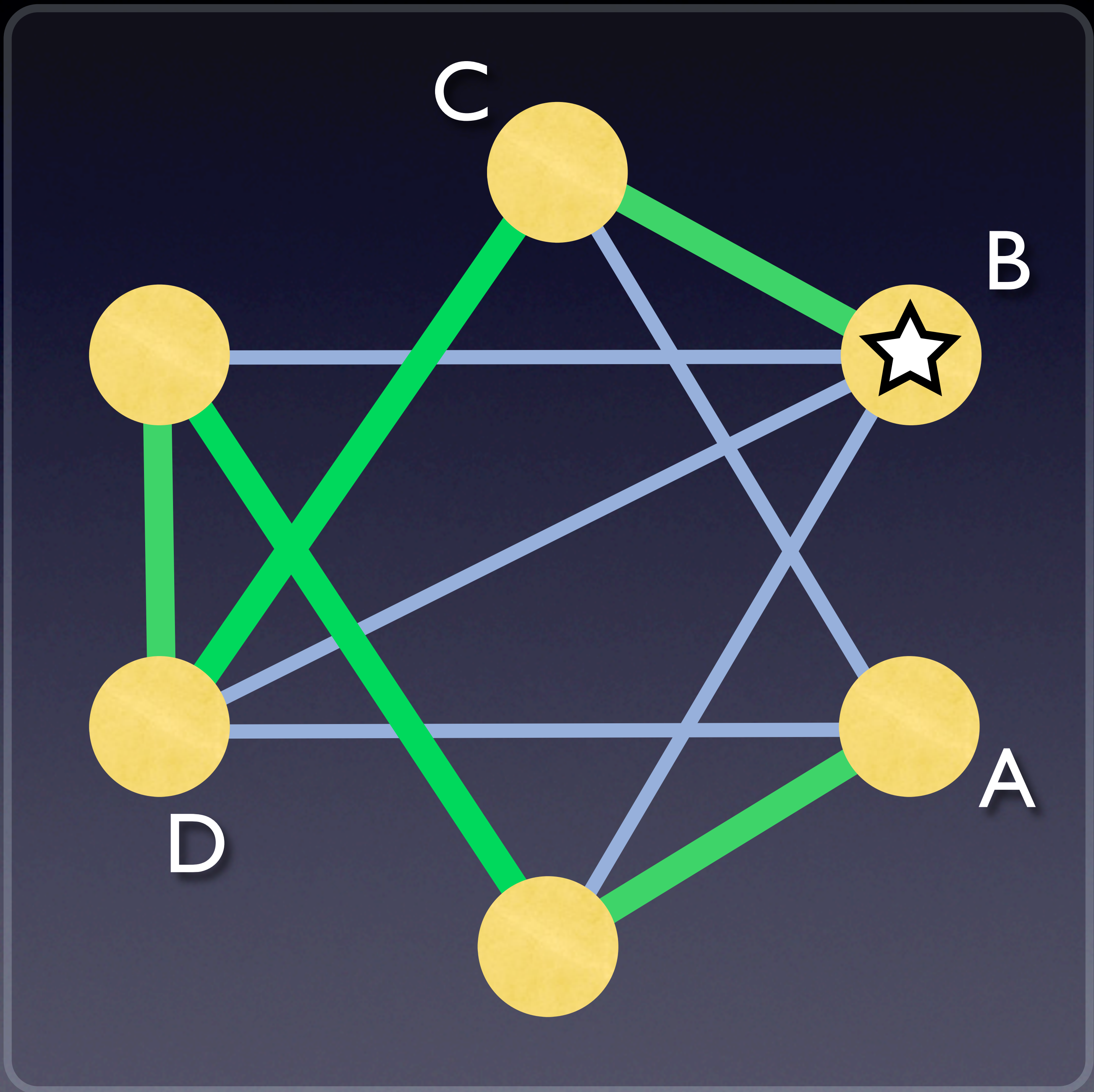

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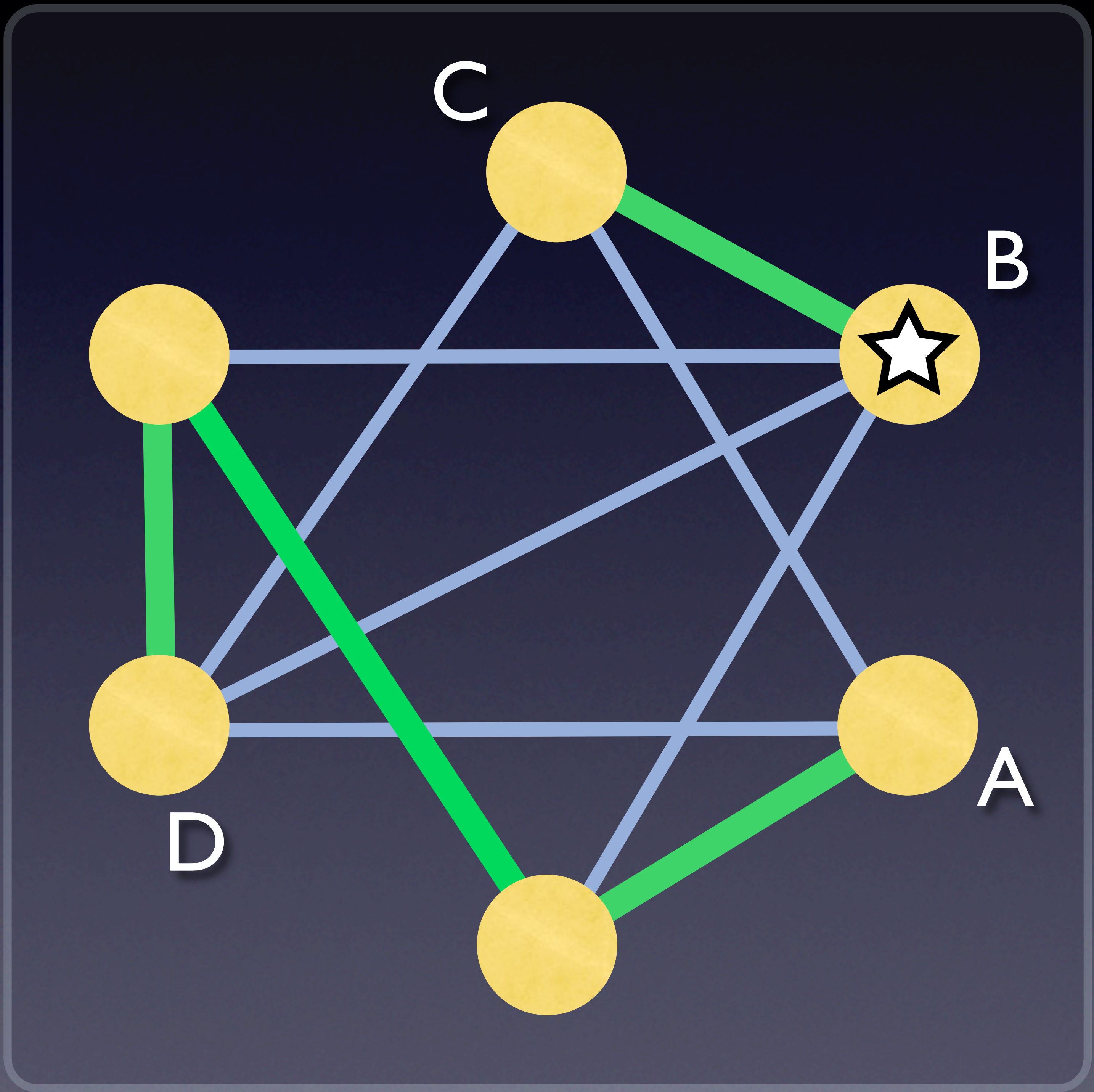

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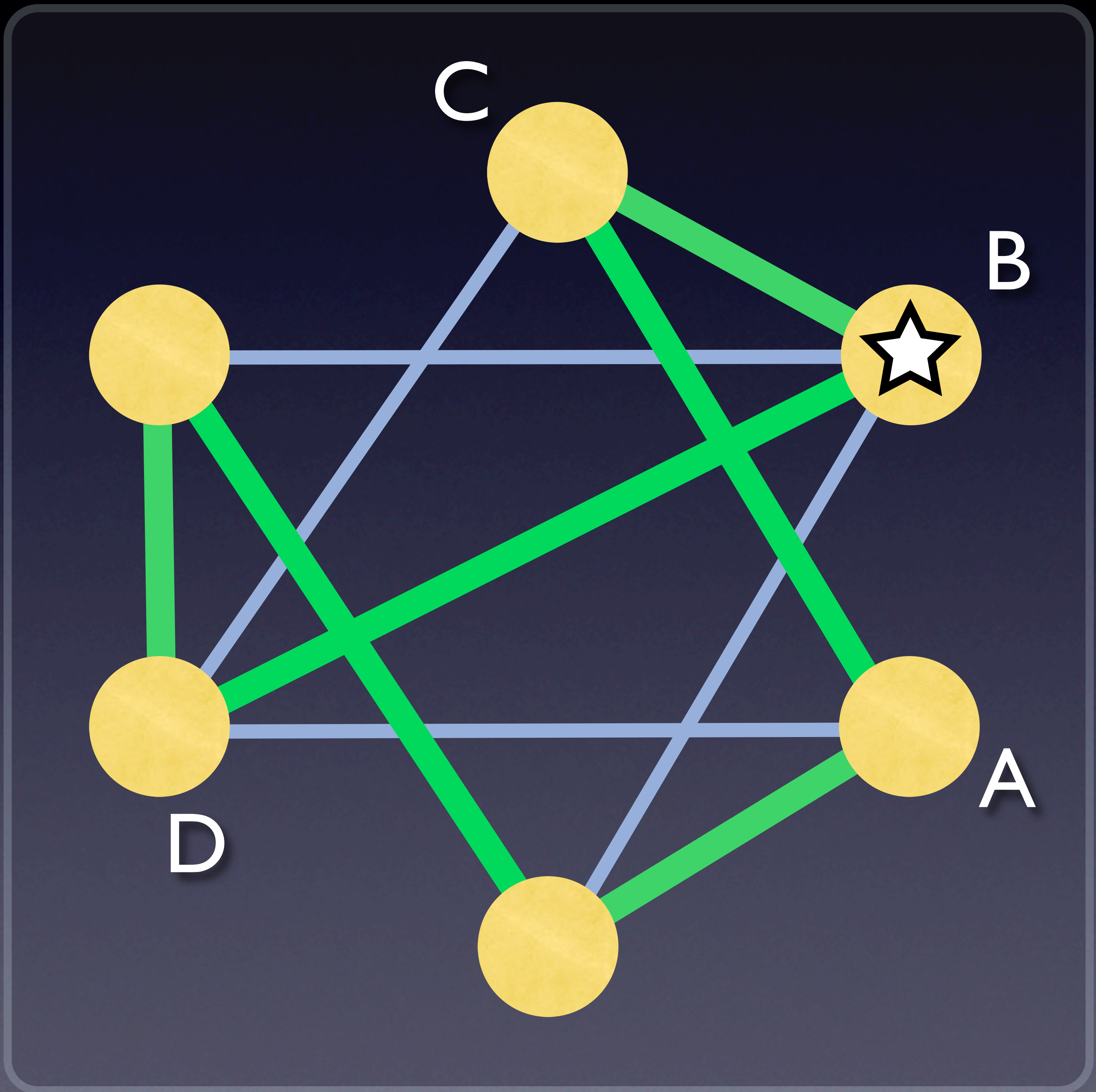

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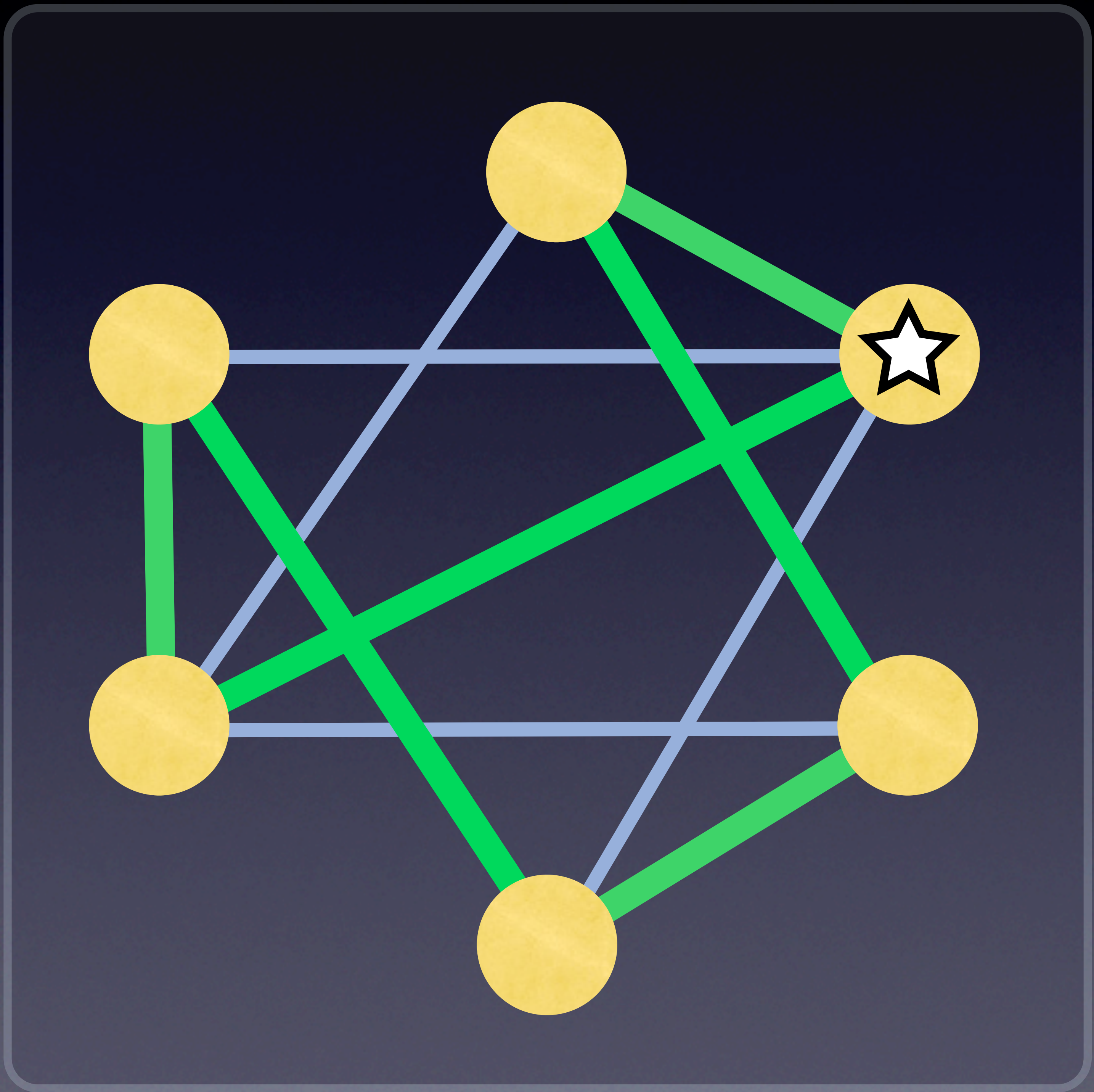

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Implementation

- The algorithm as discussed was slightly modified to use two graphs – the pretend circuit, and the true graph
- Implemented in C++ using the Boost graph library and Xcode
- Command-line only (but a GUI frontend could be constructed)

Thank you.

alanhogan.com/asu/hamiltonian-circuit

Read more about this project,
download this presentation, or
get a copy of the source code online.

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